Darboux Fundamental Theorem of Calculus

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It was well known that if a function f is differentiable on (a, b) and continuous on $[a, b]$ and if f' is Riemann integrable, then

$$\int_a^b f'(x)dx = f(b) - f(a).$$

The usual proof of this fact is to use the Mean Value Theorem. The proof is very easy following the Theorem in *Riemann Integral and Sum of Infinite Series* and using a special Riemann sum with respect to a regular partition and a choice of the points satisfying the Mean Value Theorem in each interval.

We shall give a proof without using Mean Value Theorem.

We shall restate Theorem 2 and Theorem 2' in "*Do we need Mean Value Theorem to prove* $f'(x) = 0$ *on (a, b) implies that* f = *constant on* (a, b)*?*" as follows.

Theorem 1. If $f:[a, b] \to \mathbf{R}$ is differentiable, then for any u, v in $[a, b]$ with $u < v$, there exists a point x and a point y in $[u, v]$ such that $f'(x) \geq \dfrac{f(v)-f(u)}{v-u} \geq f'(y)$ or equivalently, $f'(x)(v-u) \geq f(v)-f(u) \geq f'(y)(v-u)$.

Proof of Darboux Theorem.
We are going to make use of Theorem 1 to define two sequences of Riemann sums that both converge to the Riemann integral $\int_a^b f'(x)dx$.
For each integer $n > 1$ define the regular partition P_n

$$P_n: \ a = x_0 < x_1 < \ldots < x_n = b ,$$

with $\| P_n \| = (b-a)/n$, $x_k = a + k\dfrac{b-a}{n}$, for $k = 0, 1, \ldots, n$. For each $k = 1, \ldots, n$, by Theorem 1, there exist ξ_k and η_k in $[x_{k-1}, x_k]$ such that

$$f'(\xi_k)(x_k - x_{k-1}) \leq f(x_k) - f(x_{k-1}) \leq f'(\eta_k)(x_k - x_{k-1}).$$

Define S_n to be the Riemann sum with respect to the partition P_n witth the choice of points in each subinterval $[x_{k-1}, x_k]$ given by ξ_k . Then

$$S_n = \sum_{k=1}^{n} f'(\xi_k)(x_k - x_{k-1}) \leq \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(x_n) - f(x_0) = f(b) - f(a) .$$

Similarly, we define T_n to be the Riemann sum with respect to the partition P_n witth the choice of points in each subinterval $[x_{k-1}, x_k]$ given by η_k . Then we also have

$$T_n = \sum_{k=1}^{n} f'(\eta_k)(x_k - x_{k-1}) \geq \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(x_n) - f(x_0) = f(b) - f(a) .$$

Thus, by the Theorem in *Riemann Integral and Sum of Infinite Series*,

$$\int_a^b f'(x)dx = \lim_{n\to\infty} S_n \leq f(b) - f(a).$$

Similarly, we have $\int_a^b f'(x)dx = \lim_{n\to\infty} T_n \geq f(b) - f(a)$.
Therefore, $\int_a^b f'(x)dx = f(b) - f(a)$.