Continuity and Differentiability

Ng Tze Beng

These two notions, continuity and differentiability concerning the behaviour of a function have historically been linked. It was long thought that if a function is continuous, then it is differentiable. There might have been a number of reasons for this. One was that it was thought that function is given by a single algebraic expression. Since Weierstrass had shown a counter example to this we are now more cautious about linking notions about mathematical objects by some geometrical intuition. Now an easy counterexample would be one that is continuous but not differentiable at a single point. (See my note on differentiability of piecewise defined functions.) The more difficult and deeper examples are the following.

1. There is a function, which is continuous everywhere but not differentiable everywhere in the sense that the derivative does not exists but at an every where dense set of points the derivative at each point is $+\infty$ and at another every where dense set of points the derivative at each point is $-\infty$. (Cellérier's example: $f(x)=\sum_{n=1}^{\infty} a^{-n}\sin(a^n x)$, where a is a sufficiently large even integer.)
2. There is a function, which is continuous everywhere but not differentiable everywhere in the sense that the derivative at each point does not exist (not even in the sense of $\pm\infty$ as in example 1). This was first given by Weierstrass. His function is $f(x)=\sum_{n=0}^{\infty} a^{n}\sin(b^n \pi x)$, where b is an odd integer, a any positive real number less than 1 and $ab > 1 + (3\pi/2)$.

(An account of the construction of these and other similar functions can be found in E. W. Hobson, The theory of functions of a real variable, volume II, chapter 6. Cambridge University Press, 1926.)

So these two examples tell us that a continuous function can behave as badly as predicted.

If the function f is differentiable, what can we say about f' the derived function? Does differentiable function behave better than just continuous function? It is easy to produce example of a function f, where f is differentiable but f' is not continuous at some point.

3 There is a function f, which is differentiable but f' is discontinuous on a everywhere dense set of points and is non Riemann integrable. This example is rather complicated to construct. A construction is given in Section 276 of E. W. Hobson, The theory of functions of a real variable, volume II, chapter 6. Cambridge University Press, 1926. This dashes our hope for the possibility that a derived function would be better behaved.

But one thing is true (though not easily proven) that if f is differentiable then the derived function f' must be continuous somewhere, that is, it cannot be discontinuous everywhere. To understand this we need to know more about the points at which any function is continuous, that is, a general result concerning the set of point at which a function is continuous. For a proof of this, see B.R. Gelbaum and J M H Olmstead, Theorems and counterexamples in mathematics, page 53, Remark 2.1.2.1, Springer 1990.

On the other hand, suppose we know that the derived function f' is almost a well behaved function, can we say the same about f itself? There is a simple answer to this and also a somewhat surprising answer. The following bears testimony to this.

4 If $f: I \to \mathbf{R}$, where I is an interval, has zero derivative except perhaps on a finite number of points in I, then f is a constant function.

How much can we hope to vary or relax the condition on the derived function f'?

5 If f' is bounded and is zero except perhaps on a set which is at most countable (then f' is continuous except on a set of measure zero), then f is a constant function.

We can rephrase example 5 as follows**.**

6 Suppose (1) f' is bounded and is continuous except perhaps on a set of measure zero and (2) $f' = 0$ except perhaps on a set of measure zero. Then f is constant.
Proof. Condition (1) says that f' is Riemann integrable. Then by Darboux Theorem $\int_a^x f'(t)dt = f(x) - f(a)$. But by Condition (2) $\int_a^x f'(t)dt = \int_a^x 0dt = 0$. Therefore, $f(x) = f(a)$ for all x and so f is a constant function

This means that we truly cannot have a non constant function whose derived function f' is bounded, continuous but perhaps not on a set of measure zero and that $f' = 0$ except on a set of measure zero.

But the following example tells a different story.

7 There is a function f, which is everywhere differentiable, $f' = 0$ except on a set of measure zero but f is not only not constant but is strictly increasing. The derived function f' is actually not Riemann integrable.
An example of this is given in Example 2.1.2.1 in B.R. Gelbaum and J M H Olmstead, Theorems and counterexamples in mathematics, Springer 1990.