Concavity and Lines.

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Not all calculus text books give the same definition for concavity. Most would require differentiability. One is often asked about the equivalence of various differing definitions of concavity. Some of the observation of the equivalence is that any non vertical line can only intersect the graph of a function whose graph is concave upward or downward on an open interval in at most two points. We start with our definition 1. Then we show that under certain condition this is equivalent to some other property which is written into an equivalent definition.

Recall the following definition of concavity.

Definition 1. Suppose $f : D \to \mathbf{R}$ is a continuous function defined on an interval D. Suppose a is a point in the interior of D. Then the graph of f is *concave upward* (respectively *concave downward*) at $x = a$ if there exists a small neighbourhood N of a such that in this small neighbourhood the graph of f lies above (respectively below) the tangent line to the graph of f at $(a, f(a))$ except for the point of tangency. That is to say, the graph of f is *concave upward* (respectively *concave downward*) at $x = a$ if there exists a $\delta > 0$ such that for all x not equal to a in $(a-\delta, a+\delta)$, $f(x) > f(a) + f'(a)(x - a)$ (respectively $f(x) < f(a) + f'(a)(x - a)$). We say the graph of f is concave upward (respectively concave downward) on an open interval I if the graph of f is concave upward (respectively concave downward) at x for all x in I.

Theorem 1. Suppose the graph of a function f is either concave upward or concave downward on an open interval I. Then any tangent line to the graph of f can only intersect the graph of f at the point of tangency.

Proof. We shall prove the theorem for the case f is concave upward on the open interval I. Note that since the graph of f is concave upward on I, the function f is differentiable on I. Suppose there exists a point k in I such that the tangent line at $(k, f(k))$ meets the graph again at the point $(p, f(p))$. We may assume that $k < p$. Then the equation of the tangent line at $x = k$ is given by $y = f(k) + f'(k)(x - k)$. We now proceed by "tilting the graph". Let $g : [k, p] \to \mathbf{R}$ be defined by $g(x) = f(x) - f(k) - f'(k)(x - k)$. Then since f is differentiable on $[k, p]$ g is also differentiable on (k, p) and continuous on $[k, p]$. By the Extreme Value Theorem, there exists an absolute maximum of g on $[k, p]$. Note that $g(k) = 0$ and $g(p) = 0$ since $f(p)$ lies on the tangent line to the graph of f at $x = k$ so that $f(p) = f(k) + f'(k)(p - k)$. Since the graph of f is concave upward at $x = k$, there exists $\delta > 0$ such that for all x in $(k, k + \delta)$, $f(x) > f(k) + f'(k)(x - k)$ and so $g(x) > 0$. Hence the absolute maximum of g can only occur in the interior of $[k, p]$. Suppose that it occurs at $x = d$ in the interior of $[k, p]$. Then for all x in $[k, p]$, $g(x) = f(x) - f(k) - f'(k)(x - k) \le g(d) = f(d) - f(k) - f'(k)(d - k)$. In particular $f'(d) = f'(k)$. This is because since $g(d)$ is a relative maximum, $g'(d) = 0$ and so since $g'(x) = f'(x) - f'(k)$ for x in (k, p), $g'(d) = 0$ implies that $f'(d) = f'(k)$. Therefore, we have that for all x in $[k, p]$, $f(x) \le f(d) + f'(d)(x - d)$, which is derived from $g(x) \le g(d)$. But since the equation of the tangent line to f at the point $x = d$ is given by $y = f(d) + f'(d)(x - d)$, we therefore conclude that there exists a $\delta > 0$ such that $(d - \delta, d + \delta) \subseteq [k, p]$ and $f(x) > f(d) + f'(d)(x - d)$ for $x \ne d$ in $(d - \delta, d + \delta)$. But we have just shown

that for $x \neq d$ in $(d - \delta, d + \delta)$, $f(x) \le f(d) + f'(d)(x - d)$. This contradiction shows that the tangent line at any point k cannot meet the graph of f at $(p, f(p))$. If $p < k$, we can show similarly, that the tangent line cannot meet the graph of f at $(p, f(p))$ too. Hence any tangent line to the graph of f at any point $(x, f(x))$ cannot intersect the graph of f other than the point of tangency $(x, f(x))$.
(This argument also applies to the case when the graph of f is concave downward on the open interval I.) This completes the proof.

Theorem 2. If the function f is concave upward on an open interval I, then the derived function f' is increasing on I.

Proof. Take any two points $c < d$ in I. By Theorem 1, since the tangent line at any point on the graph can only meets the graph exactly once, and since the graph of f is concave upward on I, for any point k in I,

$$f(x) > f(k) + f'(k)(x - k) \quad \text{for } x \neq k.$$

Thus we have

$$f(x) > f(c) + f'(c)(x - c) \quad \text{for } x \neq c \text{ in } I \qquad \text{---------------} \quad (1)$$

And

$$f(x) > f(d) + f'(d)(x - d) \quad \text{for } x \neq d \text{ in } I \qquad \text{--------------------} \ (2)$$

Hence, from (1), putting $x = d$, $f(d) > f(c) + f'(c)(d - c)$ so that $\frac{f(d) - f(c)}{d - c} > f'(c)$. We also have by setting $x = c$ in (2), $f(c) > f(d) + f'(d)(c - d)$ so that $\frac{f(d) - f(c)}{d - c} < f'(d)$.
Therefore, $f'(c) < \frac{f(d) - f(c)}{d - c} < f'(d)$. This shows that f' is (strictly) increasing.

Theorem 3. If the function f is concave downward on an open interval I, then the derived function f' is decreasing on I.

The proof of Theorem 3 is similar to Theorem 2.

Theorem 4. The converse of Theorem 2 is also true. That is, if the derived function f' is increasing on an open interval I, then the graph of the function f is concave upward on an open interval I.

Proof. The proof makes use of a similar construct as in the proof of Theorem 1.
Take any point c in I. Define $g(x) = f(x) - f(c) - f'(c)(x - c)$ for any x in I. Then g is differentiable on I and $g'(x) = f'(x) - f'(c)$ for any x in I. Now for $x > c$ in I, $f'(x) > f'(c)$. Therefore, $x > c$ in I implies that $g'(x) = f'(x) - f'(c) > 0$. Therefore, g is increasing on the interval $[c, \infty) \cap I$. Now note that $g(c) = 0$. Hence we can conclude that $x > c$ in I implies that $g(x) > g(c) = 0$. This means $f(x) - f(c) - f'(c)(x - c) > 0$ for any $x > c$ in I. Hence, for any $x > c$ in I, we have

$$f(x) > f(c) + f'(c)(x - c).$$

Similarly for any $x < c$ in I, $f'(x) < f'(c)$. Therefore, for any $x < c$ in I, $g'(x) = f'(x) - f'(c) < 0$. We can now conclude that g is decreasing on $(-\infty, c] \cap I$. Therefore, for any $x < c$ in I, $g(x) > g(c) = 0$. Hence for any $x < c$ in I, $f(x) - f(c) - f'(c)(x - c) > 0$. We then have for any $x < c$ in I,

$$f(x) > f(c) + f'(c)(x - c).$$

In this way we have shown that for all $x \neq c$ in I, $f(x) > f(c) + f'(c)(x-c)$. Therefore, by definition 1, the graph of f is concave upward at $x = c$. Since this is so for any c in I, the graph of f is concave upward on I. We have proved much more. For any $x > c$ in I, $\dfrac{f(x)-f(c)}{x-c} > f'(c)$ and for any $x < c$ in I, $\dfrac{f(x)-f(c)}{x-c} < f'(c)$. The case when the derived function f' is decreasing is proven similarly.

Theorem 5. If the function f is differentiable and the derived function f' is decreasing on an open interval I, then the graph of the function f is concave downward on an open interval I.

The proof of Theorem 5 is similar to that of Theorem 4, almost word for word except for the change of inequality.

Theorem 4 and Theorem 5 set the stage for the alternative definition of concavity on open intervals.

Theorem 6. Suppose the function f is differentiable on an open interval I. Then
1. The graph of f is concave upward on I, if and only if, the derived function f' is increasing on I.
2. The graph of f is concave downward on I, if and only if, the derived function f' is decreasing on I.

Proof. Just combine Theorems 2, 3, 4 and 5.

Theorem 7. If the graph of a function defined on an open interval I is either concave upward or concave downward, no line can intersect the graph of f in more than two points.

Proof. We shall prove the case when the graph of f is concave upward on I. By Theorem 4, f' is increasing**.** Any vertical line can only intersect the graph of f in at most one point. Now assume that l is a line of gradient m. Suppose l intersects the graph of f in more than two points. That means there exist $a < b < c$ in the domain of f such that the graph of f intersects the line l at $x = a$, b and c. Then as in the proof of Theorem 4, we have $m = \dfrac{f(b)-f(a)}{b-a} < f'(b)$. We also have $f'(b) < m = \dfrac{f(c)-f(b)}{c-b}$. But then these are contradictory statements. Therefore, the line l cannot intersect the graph of f in more than two points. Similarly, we can prove the case when the graph of f is concave downward on I.

However it is possible that the graph of a function is concave upward at a point does not necessarily give information whether the derived function f' is increasing in a neighbourhood of that point.

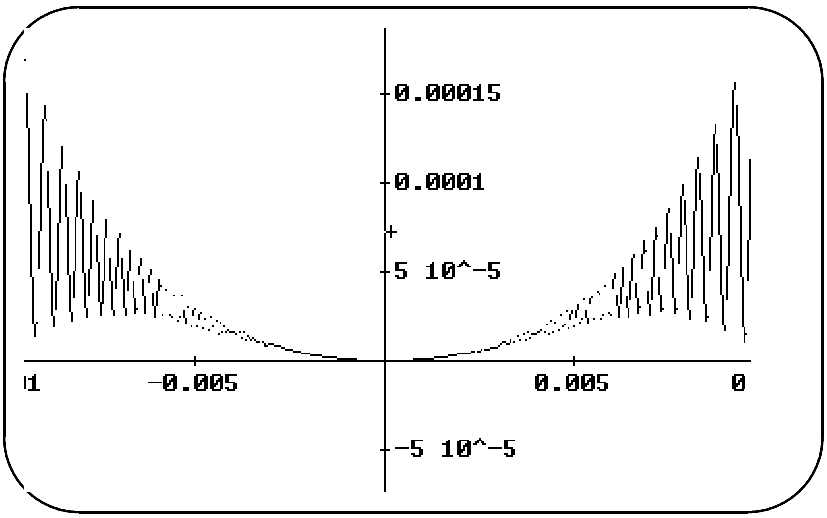
Example 8. Let $f(x) = \begin{cases} x^2 + 10000x^4 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$. Then f is differentiable on **R**, the derivative at $x = 0$, $f'(0)$ is equal to 0 by the Squeeze Theorem. The derived function is given by

$$f'(x) = \begin{cases} 2x + 40000x^3 \sin(1/x) - 10000x^2 \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and the second derived function is

$$f''(x) = \begin{cases} 2 + 120000x^2 \sin(1/x) - 60000x\cos(1/x) - 10000\sin(1/x), & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

The large constant is given here for a help to plot the graph of this function to observe the perpectual small oscillation. Note that when $x = 1/((2k+1)\pi/2)$, $\sin(1/x) = 1$ when k is even and -1 when k is odd. Thus, for any $\delta > 0$, choose integer k such that $1/((2k+1)\pi/2) < \min(\delta, 1/100)$. Let x_δ to be $1/((2k+1)\pi/2)$. Then obviously, for even k, $f''(x_\delta) = 2 + 120000x_\delta^2 - 10000 < 14 - 10000 < 0$. Then since f'' is continuous at x_δ, there exists a small open neighbourhood N_δ of x_δ in $(0, \delta)$ such that $f''(x) < 0$ for all x in this neighbourhood. Therefore f' is decreasing in N_δ. Thus, for any $\delta > 0$, we can find a neighbourhood (an interval) N_δ such that f' is decreasing in N_δ. Note that $f''(0) = 2 > 0$. Therefore, the graph of f is concave upward at the point $x = 0$. But by the above remark f' cannot be increasing in any neighbourhood containing $x = 0$. The derived function f' fails to be increasing in any open interval containing $x = 0$ simply because we can always find a subinterval on which f' is decreasing. Because we can always find arbitrarily small x_δ such that $f''(x_\delta) < 0$, we can thus find arbitrary small x_δ such that the graph of f is concave downward at $x = x_\delta$. For this function, there is no open interval containing 0 on which the function f is concave upward. Therefore, we cannot apply Theorem 6 to give any conclusion. Below is a sketch of the function.



A related question is the following.

The notion of 'increasing' is one involving a non-trivial interval. A local information like derivative of f at a point a is positive does not guarantee that the function is increasing upto and including a or *after a and including a* or *in any interval containing a. A case in point is the derived function f' in Example 8 at the point $x = 0$.*

Example 9. Consider the function $f(x) = \begin{cases} x + 4x^2\cos(\frac{1}{x}), x \neq 0 \\ 0, x = 0 \end{cases}$. Then $f'(0) = 1 > 0$ but f is neither increasing nor decreasing on any interval containing 0.

This is because for any $\delta > 0$ we can choose any integer $n > 0$ such that $\delta > 1/(2n\pi+\pi/2) > 1/(2n\pi+\pi) > 1/(2n\pi+2\pi) > 0$ but $f(1/(2n\pi+\pi)) = 1/(2n\pi+\pi) - 4/(2n\pi+\pi)^2 < 1/(2n\pi+2\pi) + 4/(2n\pi+2\pi)^2 = f(1/(2n\pi+2\pi))$ and that when $1/(2n\pi+\pi/2) > 1/(2n\pi+3\pi/2)$, $f(1/(2n\pi+\pi/2)) = 1/(2n\pi+\pi/2) > 1/(2n\pi+3\pi/2) = f(1/(2n\pi+3\pi/2))$.]

Theorem 10. Suppose $f:[a, b] \to \mathbf{R}$ is continuous and f is differentiable on (a, b). Suppose f' is strictly increasing on (a, b). If f is differentiable at $x = a$, where the derivative at $x = a$ is defined to be the right derivative at $x = a$, then f' is strictly increasing on $[a, b)$. If f is differentiable at $x = b$, where the derivative at $x = b$ is defined to be the left derivative at $x = a$, then f' is strictly increasing on $(a, b]$. Consequently, if f is differentiable at $x = a$ and at $x = b$, then f' is strictly increasing on $[a, b]$.

Proof. Suppose f' is differentiable and f' is strictly increasing on (a, b). Suppose on the contrary that f' is not strictly increasing on $[a, b)$. Then there exists c in (a, b) such that $f'(a) \geq f'(c)$. Then for any e such that $a < e < c$, $f'(a) \geq f'(c) > f'(e)$. Thus, by the intermediate value property of the derived function, (see my article *"Intermediate Value Theorem for the Derived Function"*), for any γ such that $f'(a) > \gamma > f'(e)$, there exists k such that $a < k < e$ with $f'(k) = \gamma$. Since f' is increasing on (a, b) $f'(k) = \gamma < f'(e)$ since $k < e$. But this contradicts $\gamma > f'(e)$ and so f' is strictly increasing on $[a, b)$. Similarly, we can proved that, if f is differentiable at $x = b$, then f' is strictly increasing on $(a, b]$. The last assertion is just an assertion of both statements.

A similar result below holds when f' is strictly decreasing on (a, b).

Theorem 11. Suppose $f:[a, b] \to \mathbf{R}$ is continuous and f is differentiable on (a, b). Suppose f' is strictly decreasing on (a, b). If f is differentiable at $x = a$, where the derivative at $x = a$ is defined to be the right derivative at $x = a$, then f' is strictly decreasing on $[a, b)$. If f is differentiable at $x = b$, where the derivative at $x = b$ is defined to be the left derivative at $x = a$, then f' is strictly decreasing on $(a, b]$. Consequently, if f is differentiable at $x = a$ and at $x = b$, then f' is strictly decreasing on $[a, b]$.

The proof of Theorem 11 is similar to that for Theorem 10 using the intermediate value property of the derived function.

Theorem 12. Suppose $f:[a, b] \to \mathbf{R}$ is continuous and f is differentiable on (a, b). Suppose the graph of f is concave upward on (a, b). If f is differentiable at $x = a$ (*respectively* $x = b$), the graph of f lies above the tangent line at $(a, f(a))$ (*respectively* $(b, f(b))$) except for the point of tangency.

Remark. By Theorem 1, The graph of f is above the tangent line at $(x, f(x))$ for any x in (a, b). Theorem 12 says that the same is also true of the tangent line at the *end point* whenever f is also differentiable there. It is to be emphasized that in Theoorem 12, at the end point a, it is a portion of the graph of f on the **right** of a that is above the tangent line at $(a, f(a))$ and at the other end point b, it is a portion of the graph on the **left** of b that is above the tangent line at $(b, f(b))$.

Proof of Theorem 12. By Theorem 2, f' is strictly increasing on (a, b). Suppose f is differentiable at $x = a$, then by Theorem 10, f' is strictly increasing on $[a, b)$.
Thus, by the Mean Value Theorem, for any x in $(a, b]$, i.e., $a < x \le b$, there exists c in (a, x) such that

$$\frac{f(x)-f(a)}{x-a} = f'(c) > f'(a), \quad \text{-------------} (3)$$

since f' is strictly increasing on $[a, b)$. Therefore, multiplying (3) by $(x - a) > 0$, we obtained

$$f(x) > f(a) + (x - a)f'(a).$$

This means the graph of f is above the tangent line at the point $(a, f(a))$ except for the point of tangency.
Similarly, when f is differentiable at $x = b$, we can show that for all x such that $a \le x < b$,

$$f(x) > f(b) + (x - b)f'(b).$$

A similar result stated below also holds true in the case when f is concave downward.

Theorem 13. Suppose $f:[a, b] \to \mathbf{R}$ is continuous and f is differentiable on (a, b). Suppose the graph of f is concave downward on (a, b). If f is differentiable at $x = a$ (*respectively* $x = b$), the graph of f lies below the tangent line at $(a, f(a))$ (*respectively* $(b, f(b))$) except for the point of tangency.

The proof of Theorem 13 is similar to that of Theorem 12, where we use Theorem 3 and Theorem 11 instead.

Theorem 14. Suppose $f:[a, b] \to \mathbf{R}$ is continuous. Let $x_0 \in (a, b)$. Suppose f is differentiable on (a, b) except possibly at x_0. Suppose there exists $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$ and the graph of f is concave upward on $(x_0 - \delta , x_0)$ and concave downward on $(x_0 , x_0 + \delta)$.
(1) If f is differentiable at x_0 , then

$$f(x) > f(x_0) + (x - x_0)f'(x_0) \text{ for } x_0 - \delta \le x < x_0$$

and

$$f(x) < f(x_0) + (x - x_0)f'(x_0) \text{ for } x_0 < x \le x_0 + \delta.$$

(2) If the left derivative of f at x_0 , $f_-'(x_0)$, exists, then

$$f(x) > f(x_0) + (x - x_0)f_-'(x_0) \text{ for } x_0 - \delta \le x < x_0.$$

(3) If the right derivative of f at x_0 , $f_+'(x_0)$, exists, then

$$f(x) < f(x_0) + (x - x_0)f_+'(x_0) \text{ for } x_0 < x \le x_0 + \delta.$$

Proof. We shall prove part (2) first. If $f_-'(x_0)$ exists, then f restricted to $(x_0 - \delta, x_0]$ is differentiable and part (2) follows from Theorem 12. Similarly Part (3) follows from Theorem 13. Part (1) is just part (2) and (3) put together since f is differentiable at x_0 if and only the left and right derivatives of f at x_0 exist and are the same.

Remark.
1. Note that the concavity condition implies the following consequence. If f is concave upward on $(x_0 - \delta , x_0)$ and the left derivative of at x_0 exists, then by Theorem 10, f' is strictly increasing on $(x_0 - \delta , x_0]$ and so is bounded above by the left derivative at x_0, $f_-'(x_0)$. Therefore, the limit $\lim_{x \to x_0^-} f'(x)$ exists and so by either the intermediate value property

of the derived function or L' Hôpital's Rule $\lim_{x\to x_0^-} f'(x) = f'_-(x_0)$. Conversely, if the limit $\lim_{x\to x_0^-} f'(x)$ exists, then the left derivative of f at x_0 exists and $\lim_{x\to x_0^-} f'(x) = f'_-(x_0)$. Hence, $f'_-(x_0)$ exists, if and only if, $\lim_{x\to x_0^-} f'(x)$ exists. Similarly, when f is concave downward on $(x_0, x_0 + \delta)$, $f'_+(x_0)$ exists, if and only if, $\lim_{x\to x_0^+} f'(x)$ exists and $\lim_{x\to x_0^+} f'(x) = f'_+(x_0)$.

2. Note that Theorem 14 part (1) says that the tangent line at $(x_0, f(x_0))$ crosses the graph of f there. This property is sometimes taken to be the definition of a point of inflection.

A result stated below which is similar to Theorem 14 holds true.

Theorem 15 Suppose $f:[a, b] \to \mathbf{R}$ is continuous. Let $x_0 \in (a, b)$. Suppose f is differentiable on (a, b) except possibly at x_0. Suppose there exists $\delta > 0$ such that $a < x_0 - \delta < x_0 + \delta < b$ and the graph of f is concave downward on $(x_0 - \delta, x_0)$ and concave up on $(x_0, x_0 + \delta)$.

(1) If f is differentiable at x_0, then

$$f(x) < f(x_0) + (x - x_0) f'(x_0) \quad \text{for } x_0 - \delta \le x < x_0$$

and

$$f(x) > f(x_0) + (x - x_0) f'(x_0) \quad \text{for } x_0 < x \le x_0 + \delta.$$

(2) If the left derivative of f at x_0, $f_-'(x_0)$, exists, then

$$f(x) < f(x_0) + (x - x_0) f_-'(x_0) \quad \text{for } x_0 - \delta \le x < x_0.$$

(3) If the right derivative of f at x_0, $f_+'(x_0)$, exists, then

$$f(x) > f(x_0) + (x - x_0) f_+'(x_0) \quad \text{for } x_0 < x \le x_0 + \delta.$$

The proof of Theorem 15 is exactly the same as for Theorem 14.

Recall the definition of a point of inflection below.

Definition 16. A point $(c, f(c))$ is a *point of inflection* of the graph of the function f if f is continuous at c and there is an open interval containing c such that the graph of f changes from concave upward before c to concave downward after c or from concave downward before c to concave upward after c.

Thus, when $(c, f(c))$ is a point of inflection and f is differentiable at c, then the tangent line of f at $(c, f(c))$ crosses the graph of f there by Theorem 14 and 15. Hence the function can never be concave up or concave down at the point $(c, f(c))$. Therefore, if a function f is a twice differentiable function, then at any point, where the second derivative is positive, the graph of f is concave up there, and at any point with negative second derivative there, the graph of f is concave down there. However, there does not exist a point c, where before the point *and* including the point c the graph of f is concave up and after the point c the graph of f is concave down.

Remark.

1. Suppose f is a twice differentiable function. Then for any x, $f''(x)$ is either positive, negative or zero. Therefore, we can decide the concavity of the graph of f at $(x, f(x))$ when the second derivative at x is not zero. However, at points, where the second derivative is zero, the graph of f may be concave upward, downward or neither. For instance, for the function

f, defined by $f(x) = \begin{cases} x^4 + x^6 \sin(\frac{1}{x^2}), \text{ if } x \neq 0, \\ 0 \text{ , if } x = 0 \end{cases}$, $f'(0) = f''(0) = 0$, the graph of f is concave upward at (0, 0) because for $x \neq 0$, $f(x) > 0$ but there is no interval containing 0 on which the graph of f is concave upward since its second derivative

$$f''(x) = \begin{cases} 12x^2 + 30x^4 \sin(\frac{1}{x^2}) - 18x^2 \cos(\frac{1}{x^2}) - 4\sin(\frac{1}{x^2}), \text{ if } x \neq 0, \\ 0 \text{ , if } x = 0 \end{cases}$$

can take on arbitrary sign in any interval containing 0. But the function $g(x) = x^4$ is concave upward at (0,0) and on any interval containing the point 0. By considering the function $-f$, we see that $(-f)''(0) = 0$ and the graph of $(-f)$ is concave downward at (0,0). On the other hand the function g defined by $g(x) = \begin{cases} x^4 \sin(\frac{1}{x}), \text{ if } x \neq 0, \\ 0 \text{ , if } x = 0 \end{cases}$ has second derivative given by

$$g''(x) = \begin{cases} 12x^2 \sin(\frac{1}{x}) - 6x\cos(\frac{1}{x}) - \sin(\frac{1}{x}), \text{ if } x \neq 0, \\ 0 \text{ , if } x = 0 \end{cases}.$$

Now $g'(0) = g''(0) = 0$ but the graph of g at (0, 0) is neither concave up nor concave down since $g(x)$ can be positive or negative in any interval containing 0. But such a point is not a point of inflection since there does not exists a $\delta > 0$ such that the graph of g is either concave up or concave down on $(0, \delta)$. It is difficult to say if the tangent line of g at (0, 0) which is the x-axis, 'crosses' the graph of g there without making a precise definition of a "crossing".

2. Here is a good definition of a crossing. A non-vertical line H crosses the graph of a function f at the point $(x, f(x))$ if there exists a $\delta > 0$ such that the graph of f on the interval $(x - \delta, x)$ is above the line H and on the interval $(x, x+\delta)$ the graph of f is below the line H *OR* there exists a $\delta > 0$ such that the graph of f on the interval $(x - \delta, x)$ is below the line H and on the interval $(x, x+\delta)$ the graph of f is above the line H. Thus, in view of Theorem 14 and Theorem 15 part(1), for a differentiable function f we can define a point of inflection to be a point $(x, f(x))$ where the tangent line at the point $(x, f(x))$ crosses the graph of f at the point $(x, f(x))$. This is of course the same as Definition 16 for differentiable f.