Lebesgue Integration and Composition
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The following question is of some interest. If we extend our theory of integration to the Lebesgue theory of integration, when is a bounded function Lebesgue integrable? This would involve in some sense the theory of measure. Think of this as a kind of generalization of length. Our set of domain involved must be admissible to some kind of "measure". Our next question is: Is a Lebesgue integrable function of a continuous function necessarily Lebesgue integrable? This question will be answered in the negative.

The length or measure $\lambda(I)$ of a bounded interval $I=(a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$ is $b-a$ and for unbounded interval it is defined to be $+\infty$. Let E be an arbitrary subset of **R**. Then there exists a countable cover γ of disjoint open intervals. That is γ is a countable collection of open intervals $\{I_i : i=1,\ldots\}$ such that $I_i \cap I_j = \varnothing$ for $i \neq j$ and $E \subseteq \cup \{I_i : i \in$ some index set$\}$. We define $\lambda(\gamma) = \sum_{i=1}^{\infty} \lambda(I_i)$. The length of E is defined to be $\lambda(E) =$ Infimum $\{\lambda(\gamma) : \gamma$ covers E and is a countable cover of disjoint open intervals$\}$. Then this definition is translation invariant. That is, $\lambda(E+r) = \lambda(E)$, where $E + r = \{x + r : x \in E\}$. We also have $\lambda(\varnothing) = 0$ and $\lambda(\{x\}) = 0$. λ should be non negative and has the following properties.

1. If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.
2. $\lambda\left(\bigcup_{E\in F} E\right) \leq \sum_{E\in F} \lambda(E)$ for any countable family F of subsets of **R**.

We then have that if E is a countable subset of **R**, say $E = \{x_i : i \in \mathbf{N}\}$, then
$\lambda(E) = \lambda\left(\bigcup_{i\in\mathbf{N}} \{x_i\}\right) \leq \sum_{i=1}^{\infty} \lambda(\{x_i\}) = 0$. Therefore, $\lambda(E) = 0$.

The function λ on the collection of all subsets of **R** is called the Lebesgue outer measdure. A subset E of **R** is said to be *Lebesgue measurable* if and only if for all subset X of **R**,

$$\lambda(X) = \lambda(X \cap E) + \lambda(X - E)$$

or equivalently,

$$\lambda(X) \geq \lambda(X \cap E) + \lambda(X - E).$$

Then if $E_1, E_2, \ldots, E_n$ are disjoint and Lebesgue measurable, then for any subset X of **R**,

$$\lambda\left(X \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \lambda(X \cap E_i).$$

In particular, if E_i, $i=1, 2, \ldots$ is a countable collection of Lebesgue measurable subsets, then the union $\bigcup_{i=1}^{\infty} E_i$ is also Lebesgue measurable. The following is easily observed.

3. Every subset E of **R** with $\lambda(E) = 0$ is Lebesgue measurable. Therefore, any subset of such a set is also measurable.
4. Every open subset of **R** is Lebesgue measurable. Hence any Borel subset (which is generated by open subsets) is also Lebesgue measurable.

Definition 1. A function $f: E \to \mathbf{R}$ is said to be measurable if for any open subset U of **R**, $f^{-1}(U)$ is (Lebesgue) measurable.

We have then the following theorem

Theorem 1. Suppose $f: E \to \mathbf{R}$ is a bounded function, where E is Lebesgue measurable. Then f is measurable, if and only if, the lower and upper Lebesgue integrals are the same, i.e., $\underline{\int}_E f\,d\lambda = \overline{\int}_E f\,d\lambda$, including the possibility of equaling infinity in the extended real numbers.

Suppose $f: E \to \mathbf{R}$ is a bounded function, where E is Lebesgue measurable and $\lambda(E)$ is finite, Then f is measurable, if and only if, f is Lebesgue integrable. This then reduces the question about Lebesgue integrability over a bounded interval to a question about measurability. Thus, if f is not measurable over a bounded interval, then f is not Lebesgue integrable over the same interval.

Does non measurable set exist? This depends on our system of set theory. If we admit the Axiom of Choice, then it does. If we don't admit the Axiom of Choice, then every set is Lebesgue measurable, that is, if we replace Axiom of Choice by Solovay Axiom that every $f: \mathbf{R}^n \to \mathbf{R}$ is measurable. The two systems of axioms for set theory (Zermelo-Fraenkel plus Axiom of Choice *and* Zermelo-Fraenkel plus Axiom of Solovay) are mutually incompatible although they are both consistent. The following is thus of interest to those ardent supporters of the Zermelo-Fraenkel plus Axiom of Choice.

We shall use the Axiom of Choice to define a non-measurable subset of [0,1]. Define an equivalence relation R on [0, 1] by $x\,R\,y$ if, and only if, $x - y$ is a rational number. This then partitions [0, 1] into disjoint equivalence classes. Choose a point from each of these equivalent classes to form a subset E of [0, 1]. That is, E intersects each equivalence class in exactly one point. (By the Axiom of Choice this can be done.) Then E is not Lebesgue measurable. To see this, consider the set $[0, 1] \cup -[0, 1] = [-1, 1]$. The set of rational numbers in $[-1, 1]$ is countable. Let $\{ a_n : n = 1, \ldots\}$ be an enumeration of the set of rational numbers in $[-1, 1]$. Then for each n, $E_n = \{a_n + x : x \in E\} = E + a_n$ is Lebesgue measurable if E is. Obviously, $E_n \cap E_m = \varnothing$ if $n \neq m$. This is because if $x \in E_n \cap E_m$, *then* $x = z + a_n$ for some z in E and $x = z' + a_m$ for some z' in E. Therefore, $0 = z - z' - (a_m - a_n)$ and so $z - z' = (a_m - a_n)$ and so $z\,R\,z'$. But since if $z \neq z'$, then z and z' would come from different equivalence classes and so $z\,R\,z'$ cannot hold. Thus $z = z'$ and so $a_m = a_n$ contradicting $a_m \neq a_n$. For x in [0, 1] either x is in E or $x\,R\,y$ for some y in E. Therefore, $x - y$ is a rational number in $[-1, 1]$. Thus $x - y = a_j$ for some j. Hence $x = y + a_j \in E + \{a_j\} = E_j$. We have thus shown that $[0, 1] \subseteq \bigcup_{i=1}^{\infty} E_i$. Note that each E_n is a subset of $[-1, 2]$ and so $\bigcup_{i=1}^{\infty} E_i \subseteq [-1, 2]$. Now, for each i, $\lambda(E_i) = \lambda(E + a_i) = \lambda(E)$ as λ is translation invariant. Since each E_i is measurable and $E_n \cap E_m = \varnothing$ for $n \neq m$,

$$\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \lambda(E_i) = \lim_{n \to \infty} n\lambda(E) \leq \lambda([-1, 2]) = 3.$$

This is only posible if $\lambda(E) = 0$. Therefore, $\lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$. But then we have, because $[0, 1] \subseteq \bigcup_{i=1}^{\infty} E_i$, $1 = \lambda([0,1]) \leq \lambda\left(\bigcup_{i=1}^{\infty} E_i\right) = 0$, which is absurd. Therefore, E is not Lebesgue measurable. This construction is shorter but the following is a general way of getting a non-measurable set.

Another device we would be using is the algebraic difference of two sets. This time we obtain a non-measurable subset of **R**. Define the same relation *as before but now denote by* ~ , on **R** as follows. $x \sim y$ if and only if $x - y$ is a rational number. This is obviously an equivalence relation on **R**. Denote the set of equivalence classes by **R**/~. Each equivalence class has the form

$$\{ x + r : r \in \mathbf{Q}\}.$$

Thus the set of rational numbers constitutes one class, $\{\sqrt{2} + r : r \in \mathbf{Q}\}$ is another and $\{\pi + r : r \in \mathbf{Q}\}$ is yet another. Obviously each equivalence class is countable and so since the set of real numbers **R** is uncountable, the number of equivalence classes is uncountable. This is because if the number of equivalence classes were countable then **R** being the union of countable number of equivalence classes, each of which is countable, would be countable and thus contradicts the fact that **R** is uncountable. By the Axiom of Choice, we can choose a point from each equivalence class to form an uncountable set F. We claim that this set is non-measurable. This is because the set of algebraic difference

$$F - F = \{ x - y : x, y \in F \}$$

cannot contain an interval. Because any two distinct points of F must differ by an irrational number and since F contains only one rational number, $F - F$ contains exactly one rational number namely 0. If $F - F$ were to contain an interval, it would contain rational number different from zero which is not possible. Hence by the following lemma, either F is not measurable or $\lambda(F) = 0$.

Lemma 2. If E is a Lebesgue measurable subset of **R** with positive Lebesgue measure, i.e., $\lambda(E) > 0$, then $E - E$ contains a non-trivial interval centred at the origin.

We shall prove this lemma later. Enumerate the set of rational numbers as $\{a_n : n = 1, \ldots\}$. Now define $F_n = \{a_n + x : x \in F\} = F + a_n$. Then by the definition of F, we have

$$\mathbf{R} = \bigcup_{n=1}^{\infty} F_n . \quad \text{----------------} \quad (1)$$

If $\lambda(F) = 0$, then since λ is translation invariant, $\lambda(F_n) = \lambda(F + a_n) = \lambda(F) = 0$. Thus, $\lambda(\mathbf{R}) = \lambda\left(\bigcup_{n=1}^{\infty} F_n\right) \le \sum_{n=1}^{\infty} \lambda(F_n) = 0$ implies that $\lambda(\mathbf{R}) = 0$, which is not true. Hence $\lambda(F) \neq 0$. Therefore, by Lemma 2, F is not measurable.

We have thus produced two non-Lebesgue measurable subsets, one in [0, 1] and one in **R**. We shall make use of F to produce other non-Lebesgue measurable set.

Now for the proof of Lemma 2.

Proof of Lemma 2. First we take a special open set G containing E such that

$$\lambda(G) < (1 + \varepsilon)\, \lambda(E).$$

How can we obtain G? Recall that
$\lambda(E) = \text{Inf}\{\lambda(\gamma) : \gamma$ covers E and is a countable cover of disjoint open intervals$\}$.
Now $(1 + \varepsilon)\, \lambda(E) > \lambda(E)$. Therefore, by the definition of infimum, there exists a countable cover γ of E by disjoint open intervals such that

$$(1 + \varepsilon)\, \lambda(E) > \lambda(\gamma) \ge \lambda(E).$$

Let $G = \bigcup_{I \in \gamma} I$. Then $G \supseteq E$ and $\lambda(G) = \lambda(\gamma) < (1 + \varepsilon)\lambda(E)$. We can use an enumeration of γ as $\{I_n : n = 1, \ldots\}$. Then $G = \bigcup_{n=1}^{\infty} I_n \supseteq E$. Now let $E_n = E \cap I_n$.
Then

$$E = E \cap G = E \cap \left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} E \cap I_n = \bigcup_{n=1}^{\infty} E_n.$$

It follows that each E_n is measurable since it is the intersection of two measurable sets. Note that since $\{ I_n : n = 1, \ldots\}$ is a collection of disjoint sets, $\{ E_n : n = 1, \ldots\}$ is a collection of disjoint measurable subsets. Thus,

$$\lambda(E) = \lambda\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \lambda(E_n).$$

Note that $\lambda(G) = \lambda\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} \lambda(I_n)$. Since $\lambda(G) < (1 + \varepsilon)\lambda(E)$, for some j we must have

$$\lambda(I_j) < (1 + \varepsilon)\lambda(E_j). \qquad \text{-------------------} \quad (1)$$

Let $I = I_j$ and $J = E_j$. Then $J = E_j = E \cap I_j \subseteq I_j = I$.
Take $\varepsilon = 1/3$. Then by (1), $\lambda(I) < (4/3)\lambda(J)$. That is,

$$\lambda(J) > (3/4)\lambda(I). \quad \text{-------------------------} \quad (2)$$

We now claim that if J is translated by any number d with $|d| < (1/2)\lambda(I)$, the translated set J_d has points in common with J. If this is not the case, then since $J \cup J_d \subseteq I \cup I_d$, and assuming $J \cap J_d = \varnothing$,
$2\lambda(J) = \lambda(J) + \lambda(J_d) = \lambda(J \cup J_d) \leq \lambda(I \cup I_d) \leq \lambda(I) + |d| < \lambda(I) + (1/2)\lambda(I) = (3/2)\lambda(I)$.
We would get $\lambda(J) < (3/4)\lambda(I)$ contradicting (2).
This means for some $y = x + d$ in J_d, where x is in J, y is also in J. Therefore, $d = y - x$ is in $J - J$. This is true for any d with $|d| < (1/2)\lambda(I)$ and so,

$$(-(1/2)\lambda(I), (1/2)\lambda(I)) \subseteq J - J \subseteq E - E.$$

This proves lemma 2.

Theorem 2. For any subset A of $\mathbf{R}$ with positive outer measure, i.e., $\lambda(A) > 0$, there is a non-Lebesgue measurable subset $B \subseteq A$.

Proof. Suppose $\lambda(A) > 0$. By (1), $\mathbf{R} = \bigcup_{n=1}^{\infty} F_n$ and so

$$A = A \cap \mathbf{R} = A \cap \left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{n=1}^{\infty} A \cap F_n.$$

Therefore,

$$\lambda(A) = \lambda\left(\bigcup_{n=1}^{\infty} A \cap F_n\right) \leq \sum_{n=1}^{\infty} \lambda(A \cap F_n). \quad \text{----------------} \quad (2)$$

If $A \cap F_n$ is Lebesgue measurable, then since $A \cap F_n - A \cap F_n$ does not contain a non trivial interval (because $F_n - F_n$ dose not contain a non trivial interval, a consequence of the fact that $F - F$ dose not contain a non trivial interval), by Lemma 2, $\lambda(A \cap F_n) = 0$. Therefore, since $\lambda(A) > 0$, $A \cap F_n$ cannot be Lebesgue measurable for all integer n. This is because if $A \cap F_n$ were measurable for all integer n, then by (2), $\lambda(A) \leq 0$ contradicting $\lambda(A) > 0$. Hence, for some integer j, $A \cap F_j$ is non-Lebesgue measurable. Take $B = A \cap F_j$.

Next we shall show that the Lebesgue integrable function of a continuous function neeed not be Lebesgue integrable.

Let f: [0, 1] → [0,1] be the continuous strictly increasing bijection f : [0, 1] → [0,1] defined in *Lemma 1 of Composition and Riemann integrability* mapping the Cantor set C_k of positive measure k for some $0 < k < 1$ onto the Cantor set C_0 of measure 0. Then by Theorem 2, C_k contains a non measurable subset D. Now let $K = f(D) \subseteq C_0$. Since C_0 is of measure zero, K is measurable. Thus the characteristic function χ_K is measurable, where χ_K is defined by $\chi_K(x) = 1$ if x is in K and $\chi_K(x) = 0$ if x is not in K. Therefore, χ_K is Lebesgue integrable on [0, 1]. But the composite $\chi_K \circ f = \chi_D : [0, 1] \to \mathbf{R}$ is not measurable simply because D is not measurable. Therefore, $\chi_K \circ f$ is not Lebesgue integrable.

Note that if $f : E \to \mathbf{R}$ is Lebesgue measurable and $g : B \to \mathbf{R}$ is continuous, where the range of f , $f(E) \subseteq B$, then $g \circ f$ $E \to \mathbf{R}$ is Lebesgue measurable. This is because for any open set U in $\mathbf{R}$, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is measurable. This is seen as follows. The set $g^{-1}(U)$ is open in B since g is continuous. Therefore, $f^{-1}(g^{-1}(U))$ is measurable because f is measurable. Remember that f is measurable, if and only if, for any open V , $f^{-1}(V)$ is measurable.