# The Generalized Cantor Set, Generalized Cantor Lebesgue Function, Canonical Function Mapping Cantor Set To Another, Absolute Continuity, Arc Length And Singular Functions

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Cantor sets are well known in providing counter examples in real analysis. Following my article, *The Construction of Cantor Sets*, I describe another well-known family of fat Cantor sets, i.e., Cantor sets with positive measure. It enjoys the same properties described in that article. I shall describe the construction of this family of Cantor sets, discuss their properties and the definition of the associated Cantor Lebesgue function and show that the Cantor Lebesgue function of the fat Cantor set is absolutely continuous. Related Cantor Lebesgue function, mapping one Cantor set from one family to another is described; it has similar properties as the Cantor Lebesgue function. We compute its integral, derivative and the arc length of its graph. We discuss singular functions including the singular Cantor function, strictly monotone singular functions and their characterization.

In our discussion, we start with a specific case and gradually move on to more general case by steps, first a specific family, then more general family of Cantor sets with varying ratios.

## The Cantor set $C_\gamma$

Let $0 < \gamma \leq 1$. We shall start from the closed unit interval $I = [0,1]$. At the first stage, we delete the middle open interval with length $\frac{\gamma}{3}$ from [0, 1]. We shall enumerate the open intervals to be deleted. We denote this open interval by $I(1,1)$. Then the complement of this middle interval is 2 disjoint closed interval each of length $\frac{1}{2}\left(1-\frac{\gamma}{3}\right)$. We denote the open deleted interval by $I(1,1) = (a(1,1), b(1,1))$. We denote the closed interval in the complement by $J(1,1)$ and $J(1,2)$, where the closed interval $J(1,2)$ is to the right $J(1,1)$, meaning it is ordered in such a way that every point of $J(1,2)$ is bigger than any point in $J(1,1)$.

Then at the second stage we delete the middle open interval of length $\frac{\gamma}{3^2}$ from each of the 2 remaining closed intervals. Thus, there are 2 open intervals to be deleted and they are $I(2,1) = (a(2,1), b(2,1))$ and $I(2,2) = (a(2,2), b(2,2))$. These two open intervals are ordered by the second index. That is, $I(2,2)$ is to the right of $I(2, 1)$. Hence, we are left with $4 = 2^2$ remaining closed intervals, $J(2,1)$, $J(2,2)$, $J(2,3)$ and $J(2,2^2)$ each of length

$\frac{1}{2^2}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^2\right)\right)$. Let $U(1)=I(1,1)$, $U(2)=I(2,1)\cup I(2,2)$, $G(1)=U(1)$, $G(2)=U(1)\cup U(2)$. Then $I-G(1)=J(1,1)\cup J(1,2)$, $I-G(2)=J(2,1)\cup J(2,2)\cup J(2,3)\cup J(2,2^2)$.

At stage *n* delete the middle open interval of length $\frac{\gamma}{3^n}$ from each of the $2^{n-1}$ remaining closed intervals, $J(n-1,1), J(n-1,2),\cdots,J(n-1,2^{n-1})$, each of length $\frac{1}{2^{n-1}}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^{n-1}\right)\right)$. Denote these open intervals by $I(n,1), I(n,2),\cdots,I(n,2^{n-1})$. Then this resulted in the remaining $2^n$ closed intervals, $J(n,1), J(n,2),\cdots,J(n,2^n)$, each of length $\ell_n=\frac{1}{2^n}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)$. Let $U(n)=\bigcup_{k=1}^{2^{n-1}} I(n,k)$ and $G(n)=\bigcup_{k=1}^{n} U(k)=\bigcup_{k=1}^{n}\bigcup_{j=1}^{2^{k-1}} I(k,j)$.

Observe that $I-G(n)=J(n,1)\cup J(n,2)\cup\cdots\cup J(n,2^n)$.

Note that *G*(*n*) consists of $2^n-1$ disjoint open intervals. The total length of the intervals in *G*(*n*) is given by

$$\frac{\gamma}{3}+2\frac{\gamma}{3^2}+2^2\frac{\gamma}{3^3}+\cdots+2^{n-1}\frac{\gamma}{3^n}=\frac{\gamma}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2\cdots+\left(\frac{2}{3}\right)^{n-1}\right)$$

$$=\gamma\left(1-\left(\frac{2}{3}\right)^n\right).$$

Observe that $G(n)\subseteq G(n+1)$ and $G(n+1)=G(n)\cup U(n+1)$.

Let $G=\bigcup_{k=1}^{\infty} G(k)=\bigcup_{k=1}^{\infty} U(k)$. Thus, the measure of *G* is the total length of all the *U*(*k*), that is,

$$m(G)=\lim_{n\to\infty}\gamma\left(1-\left(\frac{2}{3}\right)^n\right)=\gamma.$$

Define the generalized Cantor set $C_\gamma$ by $C_\gamma=I-G$. Hence the measure of $C_\gamma$ is given by $m(I)-m(G)=1-\gamma\geq 0$.

Note that if we take $\gamma = 1$, we would obtain the usual ternary Cantor set of measure 0. For $0 < \gamma < 1$, $m(C_\gamma)=1-\gamma>0$ and $C_\gamma$ is called the *fat Cantor set.*

The properties of $C_1$ has previously been described in *The Construction of cantor Sets*.

**Theorem 1.** The generalized Cantor set $C_\gamma$ $(0 < \gamma \le 1)$ is

(1) compact,

(2) nowhere dense, i.e., it contains no open intervals,

(3) its own boundary points,

(4) perfect, i.e., it is its own set of accumulation points,

(5) totally disconnected and

(6) between any two points in $C_\gamma$, there is an open interval not contained in $C_\gamma$.

**Proof. (1)** Since *G* is a union of open intervals, it is open. Therefore, as $C_\gamma$ is the complement of *G* in *I* and *I* is closed in $\mathbb{R}$, $C_\gamma$ is closed in $\mathbb{R}$. As $C_\gamma$ is a subset of *I*, which is bounded, $C_\gamma$ is bounded. Hence, $C_\gamma$ is compact by the Heine-Borel Theorem.

**(2)** Note that $C_\gamma = \left(\bigcup_{k=1}^{\infty} G(k)\right)^c = \bigcap_{k=1}^{\infty}\left(G(k)\right)^c$. Suppose $C_\gamma$ contains an open interval say (*c*, *d*). Then $(c,d) \subseteq \bigcap_{k=1}^{\infty}\left(G(k)\right)^c$ implies that $(c,d) \subseteq \left(G(k)\right)^c$ for each $k \ge 1$. Now each $\left(G(k)\right)^c = J(k,1) \cup J(k,2) \cup \cdots \cup J(k,2^k)$ is a disjoint union of closed intervals, each of length $\ell_k = \frac{1}{2^k}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^k\right)\right)$. As $\ell_k \to 0$, there exists an integer *N* such that $k \ge N$ implies that $0 < \ell_k < \frac{1}{2}(d-c)$. Take $k = N$. Then $(c,d) \subseteq \left(G(N)\right)^c$. Since (*c*, *d*) is connected, $(c,d) \subseteq J(N,j)$ for some $1 \le j \le 2^N$. It follows that the length of (*c*, *d*), $d - c \le \ell_N < \frac{1}{2}(d-c)$. This is absurd and so there does not exist an open interval in $C_\gamma$. This means $C_\gamma$ is nowhere dense in *I*.

**(3)** Since $C_\gamma$ is closed, the closure of $C_\gamma$ in $\mathbb{R}$ is $C_\gamma$. Let *x* be in [0, 1]. Then for any relatively open set *J*, containing *x*, say $J = U \cap [0, 1]$, where *U* is an open interval containing *x*, *J* is non-empty and contains more than one point and $J \cap G \ne \varnothing$ because $J \not\subset [0,1] - G = C_\gamma$ by part (2). This is because if $J \subseteq C_\gamma$, then the interior of *J*, which is a non-empty open interval is contained in $C_\gamma$ contradicting part (2). This means *x* is in the closure of *G* in *I*. Thus, the closure of *G* in $\mathbb{R}$, $\overline{G} = [0,1] = I$. Hence, *G* is dense in *I*. Therefore, the boundary of $C_\gamma$, $\partial C_\gamma = \overline{C_\gamma} \cap \overline{I - C_\gamma} = \overline{C_\gamma} \cap \overline{G} = C_\gamma \cap I = C_\gamma$.

**(4)** Since $C_\gamma$ is closed, the set of cluster points or limit points of $C_\gamma$ is contained in $C_\gamma$, i.e., $C_\gamma' \subseteq C_\gamma$. It remains to show that every point of $C_\gamma$ is a limit point.

Note that the end points of the closed intervals, $J(k, j_k)$, $k = 1, \ldots, \infty$, $j_k = 1, 2, \cdots, 2^k$ are in $C_\gamma$, $\ell_k = \frac{1}{2^k}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^k\right)\right)$ is the end point of $J(k, 1)$ and $1-\ell_k$ is the beginning point of $J(k, 2^k)$. As $\ell_k \to 0$ and $1-\ell_k \to 1$, 0 and 1are in $C_\gamma'$.

Suppose $C_\gamma' \neq C_\gamma$. Then there exists $x$ in $C_\gamma$ such that $x \notin C_\gamma'$. We may assume that $x \neq 0, 1$. Then there exists an open interval $(x - \delta, x + \delta)$ containing $x$ with $\delta > 0$, such that $(x-\delta, x+\delta) \subseteq (0,1)$ and $(x - \delta, x + \delta) \cap C_\gamma = \{x\}$. Thus, $(x - \delta, x + \delta) \cap ([0, 1] - C_\gamma) = (x - \delta, x) \cup (x, x + \delta)$. That means, $(x - \delta, x), (x, x + \delta) \subseteq ([0, 1] - C_\gamma) =$
$G = \bigcup_{k=1}^{\infty} U(k) = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^{k-1}} I(k, j)$ a disjoint union of open intervals. Therefore, as $(x - \delta, x)$ is connected, $(x - \delta, x) \subseteq I(k, j)$ for some $k$ and $1 \leq j \leq 2^{k-1}$, a connected component of $G$. Since $x \notin I(k, j)$, $x = \sup I(k, j) = b(k, j)$. Similarly, $(x, x+\delta) \subseteq G = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^{k-1}} I(k, j)$ implies that $(x, x + \delta) \subseteq I(n, i)$ for some $n$ and $1 \leq i \leq 2^{n-1}$. As $x \notin I(n, i)$, it follows that $x = \inf I(n, i) = a(n, i)$. Hence, $a(n, i) = b(k, j) = x$. This contradicts that $a(n, i) \neq b(k, j)$. Hence, every point of $C_\gamma$ is in $C_\gamma'$. It follows that $C_\gamma' = C_\gamma$. This means $C_\gamma$ is perfect.

**(5)** Since, by part (2), $C_\gamma$ does not contain any open interval and since the connected subsets of $\mathbb{R}$ are either the singleton sets or the intervals, the only connected components of $C_\gamma$ are the singletons $\{x\}$, $x \in C_\gamma$. Therefore, $C_\gamma$ is totally disconnected.

**(6)** Suppose $x < y$ and $x$ and $y$ are in $C_\gamma$. Then $(x, y) \cap ([0, 1] - C_\gamma) \neq \varnothing$ by part (2) and is a disjoint union of open intervals and so $(x, y)$ contains at least one open interval not in $C_\gamma$.

## The Cantor Lebesgue Function

In order to define the generalized Cantor Lebesgue function, we shall re-index the open intervals, $I(k, j_k)$, $k = 1, \ldots, \infty$, $j_k = 1, 2, \cdots, 2^{k-1}$ to reflect the values that the function will take on these intervals. $I(1,1) = I(1/2)$, $I(2,1) = I(1/4), I(2,2) = I(1/2 + 2/4) = I(3/4)$.

We show that $G(n)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\le m\le 2^n-1\right\}$ by induction. Recall
$G(n+1)=G(n)\cup U(n+1)$.

Plainly, $G(1)=U(1)=I\left(\frac{1}{2}\right)$, $G(2)=\bigcup\left\{I\left(\frac{m}{2^2}\right):1\le m\le 2^2-1\right\}$.

Assume that $G(n)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\le m\le 2^n-1\right\}$.

We re-index $U(n)=\bigcup_{k=1}^{2^{n-1}} I(n,k)$, by $I(n,k)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)=I\left(\frac{2k-1}{2^n}\right), 1\le k\le 2^{n-1}$. Then the length of $I(n,k)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$ is equal to $\frac{\gamma}{3^n}$. The index reflects the value that the Cantor Lebesgue function will take on these open intervals. That is the function will take the value $r$ in $I(r)$.

It follows that $G(n+1)=G(n)\cup U(n+1)=\bigcup\left\{I\left(\frac{m}{2^{n+1}}\right):1\le m\le 2^{n+1}-1\right\}$.

Observe that the ordering

$$0<\frac{1}{2^{n+1}}<\frac{1}{2^n}<\frac{1}{2^n}+\frac{1}{2^{n+1}}<\frac{2}{2^n}<\frac{2}{2^n}+\frac{1}{2^{n+1}}<\frac{3}{2^n}<\frac{3}{2^n}+\frac{1}{2^{n+1}}<\cdots<\frac{2^n-1}{2^n}<\frac{2^n-1}{2^n}+\frac{1}{2^{n+1}}=\frac{2^{n+1}-1}{2^{n+1}}$$

is in one to one correspondence with the ordering of the disjoint open intervals in $G(n+1)$.

Thus, by induction on $n$, $G(n)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\le m\le 2^n-1\right\}$ for all integer $n\ge 1$.

We now define $f_n:G(n)\to[0,1]$ by $f_n(x)=r$ if $x\in I(r)$ for some $r=\frac{m}{2^n}$ and $1\le m\le 2^n-1$.

Letting $n$ tend to infinity, this defines $f:G\to[0,1]$. Plainly, $f$ is constant on each of the open interval in $G$. Since each $f_n$ is an increasing function on $G(n)$, and the indexing preserves the ordering of the open intervals in $G(n)$, $f$ is also an increasing function on $G$.

**Proposition 2**. $f:G\to[0,1]$ is uniformly continuous.

Before we prove this, we state a result that we need in its proof below.

**Lemma 3.** For any $x$ and $y$ in $G$, $|x-y|\le \ell_n \Rightarrow |f(x)-f(y)|<\frac{1}{2^n}$.

**Proof.**

Let $x, y \in G$. Suppose $|x-y| \le \ell_n$. Then $x$ and $y$ cannot belong to two distinct open intervals in $G(n)$. This is because the distance between any two consecutive open intervals in $G(n)$ is $\ell_n$ and $x, y \notin G^c$. Consequently, either $x$ and $y$ belong to the same open interval, $I(r)$, in $G(n)$ or *only one* of $x$ or $y$ belongs to some open interval, $I(r)$, in $G(n)$ or $x, y \in \left(G(n)\right)^c$.

If $x$ and $y$ belong to the same open interval $I(r)$ in $G(n)$, then $\left|f(x)-f(y)\right| = r - r = 0 \le \dfrac{1}{2^n}$.

Suppose $x<y$, $x \in I\left(\dfrac{m}{2^n}\right) \subseteq G(n)$ for some $1 \le m \le 2^n - 1$ and $y \notin I\left(\dfrac{m}{2^n}\right)$. Since $|x-y| \le \ell_n$, $y$ must belong to the adjacent closed interval $J(n, m+1)$ next to and after $I\left(\dfrac{m}{2^n}\right)$. Thus,

$$\left|f(y)-f(x)\right| = f(y) - \frac{m}{2^n} < \frac{m+1}{2^n} - \frac{m}{2^n} = \frac{1}{2^n}.$$

Suppose $x<y$, $y \in I\left(\dfrac{m}{2^n}\right) \subseteq G(n)$ for some $1 \le m \le 2^n - 1$ and $x \notin I\left(\dfrac{m}{2^n}\right)$. As $|x-y| \le \ell_n$, $x$ must belong to the adjacent closed interval $J(n, m)$ before $I\left(\dfrac{m}{2^n}\right)$. Thus,

$$\left|f(y)-f(x)\right| = \frac{m}{2^n} - f(x) < \frac{m}{2^n} - \frac{m-1}{2^n} = \frac{1}{2^n}.$$

Suppose $x<y$ and $x$ and $y$ belong to the same closed interval $J(n, m)$ for some $1 \le m \le 2^n - 1$. Then for some $k > n$, $x$ and $y$ belong to $G(k)$. Consequently, $x$ and $y$ must belong to some open intervals, $I\left(\dfrac{p}{2^k}\right)$ and $I\left(\dfrac{q}{2^k}\right)$ with $\dfrac{m-1}{2^n} < \dfrac{p}{2^k}, \dfrac{q}{2^k} < \dfrac{m}{2^n}$. Hence,

$$\left|f(y)-f(x)\right| = f(y) - f(x) < \frac{m}{2^n} - \frac{m-1}{2^n} = \frac{1}{2^n}.$$

We are left with the possibility, $x<y$, $x \in J\left(n, m\right)$ and $y \in J\left(n, m+1\right)$, with $I\left(\dfrac{m}{2^n}\right)$ sited between the two consecutive closed intervals, $J\left(n, m\right)$ and $J\left(n, m+1\right)$. Note that the length of $I\left(\dfrac{m}{2^n}\right)$ is equal to $\dfrac{\gamma}{3^k}$, for some $k$ such that $1 \le k \le n$. Then $\dfrac{\gamma}{3^k} < \ell_n$. This is because $x \in J\left(n, m\right)$ and $y \in J\left(n, m+1\right)$ (and $x, y \in G$) implies that $y - x$ is great than the length of $I\left(\dfrac{m}{2^n}\right)$, which is $\dfrac{\gamma}{3^k}$. Hence, $\dfrac{\gamma}{3^k} < y - x \le \ell_n$.

Let $P = \ell_n - \frac{\gamma}{3^k} > 0$. Let $I\left(\frac{m}{2^n}\right) = \left(c(n,m), d(n,m)\right)$ and if $m \geq 2$,

$I\left(\frac{m-1}{2^n}\right) = \left(c(n,m-1), d(n,m-1)\right)$. Note that $d(n,m) - c(n,m) = \frac{\gamma}{3^k}$ and

$J\left(n,m\right) = \left[d(n,m-1), c(n,m)\right]$ if $m \geq 2$ and $J\left(n,m\right) = \left[0, c(n,1)\right]$ if $m = 1$. $x \in J\left(n,m\right)$ implies that $d(n,m-1) \leq x \leq c(n,m)$ if $m \geq 2$ and $0 \leq x \leq c(n,1)$ if $m = 1$. We let $d(n,0) = 0$.

We claim that $P > c(n,m) - x$. This is because if $P \leq c(n,m) - x$, then

$(c(n,m) - x) + \frac{\gamma}{3^k} \geq \ell_n - \frac{\gamma}{3^k} + \frac{\gamma}{3^k} = \ell_n$. Since $y > d(n,m)$ because $y$ is in $J\left(n,m+1\right)$,

$y - x > d(n,m) - x = d(n,m) - c(n,m) + c(n,m) - x = \frac{\gamma}{3^k} + c(n,m) - x \geq \ell_n$. This implies

$y - x > \ell_n$ and contradicts that $y - x \leq \ell_n$. Thus $P > c(n,m) - x$.

Hence, $x - d(n,m-1) = c(n,m) - d(n,m-1) - (c(n,m) - x)$

$$= \ell_n - (c(n,m) - x) > \ell_n - P = \frac{\lambda}{3^k}.$$

Therefore, $x > d(n,m-1) + \frac{\gamma}{3^k}$. Let $C = d(n,m-1) + \frac{\gamma}{3^k}$.

Next, we claim that $y - d(n,m) \leq x - C$.

Suppose on the contrary, $y - d(n,m) > x - C$.

Then

$$y - x = y - d(n,m) + (d(n,m) - c(n,m)) + \left(c(n,m) - x\right) = y - d(n,m) + \frac{\gamma}{3^k} + \left(c(n,m) - x\right)$$

$$> x - C + \frac{\gamma}{3^k} + \left(c(n,m) - x\right) = c(n,m) - d(n,m-1) = \ell_n.$$

This contradicts $y - x \leq \ell_n$ and so $y - d(n,m) \leq x - C$.

Therefore, $x - d(n,m-1) - \frac{\gamma}{3^k} \geq y - d(n,m)$. Hence, $x - d(n,m-1) \geq y - d(n,m) + \frac{\gamma}{3^k}$.

In the construction of the cantor set $C_\gamma$, the procedure that continues in $J\left(n,m\right)$ and $J\left(n,m+1\right)$ are exactly the same and so $f$ on $J\left(n,m\right)$ and $f$ on $J\left(n,m+1\right)$, after levelling the values on each of the closed intervals, are the same, i.e.,

$$f\left(d(n,m-1) + h\right) - \frac{m-1}{2^n} = f\left(d(n,m) + h\right) - \frac{m}{2^n} \quad \text{for} \quad 0 \leq h \leq \ell_n.$$

Therefore,

$$f\left(d(n,m-1) + x - d(n,m-1)\right) - \frac{m-1}{2^n} \geq f\left(d(n,m-1) + y - d(n,m) + \frac{\gamma}{3^k}\right) - \frac{m-1}{2^n}$$

But

$$f\left(d(n,m-1)+y-d(n,m)+\frac{\gamma}{3^k}\right)-\frac{m-1}{2^n}=f\left(d(n,m)+y-d(n,m)+\frac{\gamma}{3^k}\right)-\frac{m}{2^n}$$

$$>f\big(d(n,m)+y-d(n,m)\big)-\frac{m}{2^n}=f(y)-\frac{m}{2^n}.$$

Consequently, $f(x)-\frac{m-1}{2^n}>f(y)-\frac{m}{2^n}$. Therefore, $f(y)-f(x)<\frac{m}{2^n}-\frac{m-1}{2^n}=\frac{1}{2^n}$.

This completes the proof.

**Proof of Proposition 2.**

Given any $\varepsilon > 0$, as $\frac{1}{2^n}\to 0$, there exists a positive integer *N* such that $n\geq N\Rightarrow\frac{1}{2^n}<\varepsilon$.

Let $\delta=\ell_N$. Then by Lemma 3, for any *x, y* in *G*,

$$|x-y|\leq\delta=\ell_N\Rightarrow|f(x)-f(y)|<\frac{1}{2^N}<\varepsilon.$$

This means $f:G\to[0,1]$ is uniformly continuous on *G*.

## Definition of Cantor Lebesgue Function

Since *G* is dense in *I*, we now extend the function $f$ to all of *I*.

That this can be done is because of the following result.

*If a function g is uniformly continuous on a dense subset of a subset E in* $\mathbb{R}$, *then it can be extended to a continuous function on the whole of E.*

However, we shall give an explicit definition of the extension for $f$, which is reminiscent of the proof of the above result.

We have already shown that $G=\bigcup_{k=1}^{\infty}U(k)$ is dense in *I*.

Take any *x* in $I-G=G^c=C_\gamma$. Since *G* is dense in *I*, there exists a sequence $(x_n)$ in *G* such that $x_n\to x$. Then $(x_n)$ is a Cauchy sequence in *G*. Next, we show that $(f(x_n))$ is also a Cauchy sequence.

Now, given any $\varepsilon > 0$, there exists a positive integer *N* such that $n\geq N\Rightarrow\frac{1}{2^n}<\varepsilon$. Since $(x_n)$ is a Cauchy sequence, there exists a positive integer *M* such that $n,m\geq M\Rightarrow|x_n-x_m|<\ell_N$. Therefore, by Lemma 3,

$$n,m \geq M \Rightarrow |x_n - x_m| < \ell_N \Rightarrow |f(x_n) - f(y_m)| < \frac{1}{2^N} < \varepsilon .$$

This means that $\left(f\left(x_n\right)\right)$ is also a Cauchy sequence. Therefore, $\left(f\left(x_n\right)\right)$ is convergent by the Cauchy principle of convergence. Let $f\left(x_n\right) \to y$. Define $f\left(x\right) = y$. This is well defined. Suppose there is another sequence $\left(y_n\right)$ in *G* such that $y_n \to x$. Then $x_n - y_n \to 0$.

Thus, there exist a positive integer *L* such that

$$n \geq L \Rightarrow |x_n - y_n| < \ell_{N+1} \Rightarrow \left|f\left(x_n\right) - f\left(y_n\right)\right| < \frac{1}{2^{N+1}} < \frac{\varepsilon}{2} \text{ . -------------- (1)}$$

Since $f\left(x_n\right) \to y$, there exists integer $L_1$ such that

$$n \geq L_1 \Rightarrow \left|f\left(x_n\right) - y\right| < \frac{\varepsilon}{2} \text{. ---------------------------------- (2)}$$

Therefore, it follows from (1) and (2) that

$$n \geq \max(L, L_1) \Rightarrow \left|f\left(y_n\right) - y\right| \leq \left|f\left(y_n\right) - f\left(x_n\right)\right| + \left|f\left(x_n\right) - y\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

This means $f\left(y_n\right) \to y$. Hence, $f\left(x\right)$ is well defined.

We define $f\left(x\right)$ for every *x* in $I - G = G^c = C_\gamma$ in exactly the same way. We have thus defined a function, $f_{C_\gamma} : [0,1] \to [0,1]$, such that $f_{C_\gamma}(x) = f(x)$ for every *x* in *G*. This is the generalized Cantor Lebesgue function.

Observe that $f_{C_\gamma}(0) = 0$ and $f_{C_\gamma}(1) = 1$.

## Properties of the Cantor Lebesgue Function

**Proposition 4.** The generalized Cantor Lebesgue function, $f_{C_\gamma}$, is increasing and maps $C_\gamma$ onto $I = [0,1]$. Consequently, $C_\gamma$ is uncountable.

**Proof.**

If $x < y$ and $x, y \in G$, then $f_{C_\gamma}(x) = f(x) \leq f(y) = f_{C_\gamma}(y)$ as $f$ is increasing on *G*.

If $x < y, y \in G$ and $x \notin G$, then $y \in I(r) \subseteq G$ for some dyadic rational *r*. If $x = \inf\{k : k \in I(r)\}$, then $f_{C_\gamma}(x) = r = f(y)$, If $x < \inf\{k : k \in I(r)\}$, then take an open interval $(x-\delta, x+\delta)$, where $\delta = \frac{1}{2}(\inf(I(r)) - x) > 0$. Then $G \cap (x-\delta, x+\delta) \neq \varnothing$ since $C_\gamma$ is nowhere dense. Hence for any sequence $(x_n)$ in $G$ such that $x_n \to x$, there exists an integer $N$ such that $n \geq N \Rightarrow x_n \in G \cap (x-\delta, x+\delta)$. Since all the open intervals $I(k)$ in $G \cap (x-\delta, x+\delta) \neq \varnothing$ are to the left of $I(r)$, $f(x_n) < r = f_{C_\gamma}(y)$ for $n \geq N$ and so $f_{C_\gamma}(x) = \lim_{n\to\infty} f(x_n) \leq r = f_{C_\gamma}(y)$.

Similarly, if $x < y$, $x \in G$ and $y \notin G$, then $x \in I(r) \subseteq G$ for some dyadic rational $r$. If $y = \sup\{k : k \in I(r)\}$, then $f_{C_\gamma}(y) = r = f(x)$. If $y > \sup\{k : k \in I(r)\}$, then take an open interval $(y-\delta, y+\delta)$, where $\delta = \frac{1}{2}(y - \sup(I(r))) > 0$. Then $G \cap (y-\delta, x+y) \neq \varnothing$ since $C_\gamma$ is nowhere dense. Hence for any sequence $(y_n)$ in $G$ such that $y_n \to x$, there exists an integer $N$ such that $n \geq N \Rightarrow y_n \in G \cap (y-\delta, y+\delta)$. Since all the open intervals $I(k)$ in $G \cap (y-\delta, y+\delta) \neq \varnothing$ are to the right of $I(r)$, $f(y_n) > r = f_{C_\gamma}(x)$ for $n \geq N$ and so $f_{C_\gamma}(y) = \lim_{n\to\infty} f(y_n) \geq r = f_{C_\gamma}(x)$.

Suppose now $x < y$, and $x, y \notin G$. Then by theorem 1 part (6), there exists an open interval between $x$ and $y$ not in $C_\gamma$. Therefore, there exists z in G such that $x < z < y$. Hence, by what we have just shown, $f_{C_\gamma}(x) \leq f(z) = f_{C_\gamma}(z) \leq f_{C_\gamma}(y)$.

We have thus shown that the function, $f_{C_\gamma}$, is increasing on $I$.

Every real number $y > 0$ in [0, 1] has a non-terminating binary representation of the form $\sum_{k=1}^{\infty} \frac{b_k}{2^k}$, where $b_k = 0$ or $1$. Then the partial sum $r_n = \sum_{k=1}^{n} \frac{b_k}{2^k} \to y$. Now end points of $I(r_n)$ is in $C_\gamma$. Let $x_n = \inf I(r_n)$. Then since $(r_n)$ is strictly increasing, $(x_n)$ is also strictly increasing and since it is bounded above $(x_n)$ is convergent. Let $x_n \to x$. Then $x$ is a limit point of $C_\gamma$. Therefore, by Theorem 1 part (4), $x \in C_\gamma$. Let $y_n = \sup I(r_n)$. Then since $(r_n)$ is strictly increasing, $(y_n)$ is also strictly increasing and since it is bounded above $(y_n)$ is convergent. Let $y_n \to h$.

Since $\sum_{k=1}^{\infty} \frac{b_k}{2^k}$ is a non-terminating sequence, there exists a constant subsequence $(b_{n_j})$ of $(b_n)$ such that $b_{n_j} = 1$ for all $j \geq 1$. Therefore, $x_{n_j} \to x$ and $y_{n_j} \to h$. But the length of $I(r_{n_j})$ is equal to $\frac{1}{3^{n_j}}$, which tends to 0. Thus, $x = h$. Now take any $a_{n_j}$ in $I(r_{n_j})$. By the Squeeze

Theorem, $a_{n_j} \to x$. Therefore, $f_{C_\gamma}(x) = \lim_{j\to\infty} f(a_{n_j}) = \lim_{j\to\infty} r_{n_j} = y$. As $f_{C_\gamma}(0) = 0$, this shows that $f_{C_\gamma}$ maps the Cantor set onto [0, 1]. Hence, $f_{C_\gamma}$ maps [0, 1] onto [0, 1] and the Cantor set $C_\gamma$ is uncountable, since [0,1] is uncountable.

**Proposition 5.** The generalized Lebesgue Cantor function, $f_{C_\gamma}$, is continuous on $I = [0,1]$.

**Proof.** Since $f_{C_\gamma}$ is monotonic increasing and the image of $f_{C_\gamma}$ is an interval, $f_{C_\gamma}$ is continuous. This is because $f_{C_\gamma}$ can have only jump discontinuities, but since the image is a connected interval, no jump discontinuity is possible and so $f_{C_\gamma}$ is continuous.

**Proposition 6.** The generalized Lebesgue Cantor function, $f_{C_\gamma}:[0,1]\to[0,1]$, for the fat Cantor set, $C_\gamma$, $0 < \gamma < 1$, is Lipschitz on [0,1] with constant $\dfrac{1}{1-\gamma}$ and so it is absolutely continuous on [0,1].

**Proof.** For this we are going to use $G(n)$ to construct a polygonal approximation to $f_{C_\gamma}$. Define $f_n : I \to \mathbb{R}$, by $f_n(x) = f_{C_\gamma}(x)$ for $x$ in $G(n)$. Now $G(n) = \bigcup\left\{ I\left(\frac{m}{2^n}\right) : 1 \le m \le 2^n - 1\right\}$ and so $f_n(x) = \frac{m}{2^n}$ for $x \in I\left(\frac{m}{2^n}\right)$, for $1 \le m \le 2^n - 1$.

Define $f_n(x) = \frac{m}{2^n}$ for $x$ = end points of $I\left(\frac{m}{2^n}\right)$. Now $I - G(n) = \left(G(n)\right)^c = \bigcup_{k=1}^{2^n} J(n,k)$ is a disjoint union of closed intervals, each of length $\ell_n = \frac{1}{2^n}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)$. We define $f_n$ on each $J(n,k)$, $1 \le k \le 2^n$ to be given by the linear function or line joining the points on the graph of $f_{C_\gamma}$ given by the image of the end points of $J(n,k)$. Hence, the linear function on each of the closed interval $J(n,k)$ has the same gradient given by

$$\frac{\frac{1}{2^n}}{\ell_n} = \frac{1}{1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)} .$$

It follows that for any $x \ne y$ in $I$, $\dfrac{f_n(x)-f_n(y)}{x-y} \le \dfrac{1}{1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)}$. This means that for any $x$ and $y$ in $I$,

$$\left|f_n(x)-f_n(y)\right| \le \frac{1}{1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)}\left|x-y\right| . \text{ ------------------------- } (3)$$

Plainly, $\left|f_n(x) - f_{n+1}(x)\right| \le \dfrac{1}{2^n}$ for all $x$ in $I$. This means the sequence of function $\left(f_n(x)\right)$ converges uniformly to a continuous function, $g$, on $I$ such that $g(x) = f_{C_\gamma}(x)$ for all $x$ in $G$. It follows that $g = f_{C_\gamma}$ identically on $I$ since $G$ is dense in $I$. Therefore, by taking limit as $n$ tends to infinity in (3) we get, for any $x$, $y$ in $I$,

$$\left|f_{C_\gamma}(x) - f_{C_\gamma}(y)\right| \le \frac{1}{1-\gamma}\left|x-y\right| . \qquad \text{---------------- (4)}$$

This means $f_{C_\gamma}$ is Lipschitz on $I$ and so is absolutely continuous on $I$.

**Remark.** Absolute continuity implies continuity. A consequence of Proposition 6 is that $f_{C_\gamma}$ is continuous. Since $f_{C_\gamma}$ is increasing, $f_{C_\gamma}(0) = 0$ and $f_{C_\gamma}(1) = 1$, by the Intermediate Value Theorem, $f_{C_\gamma}$ is onto. So, for the fat Cantor set, we need not use the fact that $f_{C_\gamma}(C_\gamma) = [0,1]$ to conclude that $f_{C_\gamma}$ is onto.

**Proposition 7.** The arc length of the graph of the generalized Cantor Lebesgue function, $f_{C_\gamma} : [0,1] \to [0,1]$, for $C_\gamma$, is $\gamma + \sqrt{1+(1-\gamma)^2}$ .

**Proof.** The arc length of the graph of the polygonal approximation $f_n$ is given by the total length of the horizontal line given by the total length of $G(n)$, which is equal to $\gamma\left(1-\left(\dfrac{2}{3}\right)^n\right)$ plus the total length of the $2^n$ lines on the graph of $f_n$ joining the two points on the graph given by the end points of each of the closed intervals $J(n, k)$. Therefore, the arc length of the polygonal approximation $f_n$, is given by

$$\gamma\left(1-\left(\frac{2}{3}\right)^n\right) + 2^n\sqrt{\left(\ell_n\right)^2 + \frac{1}{2^{2n}}} = \gamma\left(1-\left(\frac{2}{3}\right)^n\right) + \sqrt{\left(2^n \ell_n\right)^2 + 1}$$

$$= \gamma\left(1-\left(\frac{2}{3}\right)^n\right) + \sqrt{\left(1-\gamma\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2 + 1} .$$

Therefore, the arc length of the graph of $f$ is given by

$$\lim_{n\to\infty}\left(\gamma\left(1-\left(\frac{2}{3}\right)^n\right) + \sqrt{\left(1-\gamma\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2 + 1}\right) = \gamma + \sqrt{1+(1-\gamma)^2} .$$

**Proposition 8.** The generalized Cantor Lebesgue function, $f_{C_\gamma}$, for the Cantor set $C_\gamma$, $0 < \gamma \leq 1$, is differentiable almost everywhere on $I$. The function $f_{C_\gamma}$ is differentiable on $G = I - C_\gamma$ and $f_{C_\gamma}'(x) = 0$ in $I - C_\gamma$. For the Cantor Lebesgue function, $f_{C_\gamma}$, for the fat Cantor set $C_\gamma$, $0 < \gamma < 1$, $f_{C_\gamma}'(x) = \dfrac{1}{1-\gamma}$ almost everywhere on $C_\gamma$. For the case of the ternary Cantor Lebesgue function, $f_{C_\gamma}$, i.e., when $\gamma = 1$, $f_{C_\gamma}'(x) = 0$ almost everywhere on [0,1].

**Proof.** As the Cantor Lebesgue function, $f_{C_\gamma}$, is monotone increasing, $f_{C_\gamma}$ is differentiable almost everywhere on $I$. Plainly, since $f_{C_\gamma}$ is constant on each of the open interval in $G = I - C_\gamma$, $f_{C_\gamma}'(x) = 0$ in $I - C_\gamma$. Obviously, if $\gamma = 1$, $f_{C_\gamma}'(x) = 0$ almost everywhere on [0,1], since $m(C_1) = 0$.

Now for the Cantor Lebesgue function, $f_{C_\gamma}$, for the fat Cantor set $C_\gamma$, $0 < \gamma < 1$, as $f_{C_\gamma}$ is monotone increasing and absolutely continuous, for any measurable subset $E$ in [0, 1],

$$\int_E f_{C_\gamma}'(x)dx = m\left(f_{C_\gamma}(E)\right), \qquad \text{------------------------ (5)}$$

where $m$ is the Lebesgue measure on $\mathbb{R}$. (See Theorem 11, *Functions of Bounded Variation and Johnson's Indicatrix*.)

Note that in general, a function $h$ defined on $I$ is absolutely continuous implies that $h$ is of bounded variation, $h$ is differentiable almost everywhere on $I$ and that $h'$ is Lebesgue integrable. In this case $f_{C_\gamma}$ is increasing implies $f_{C_\gamma}$ is differentiable almost everywhere and $f_{C_\gamma}'$ is Lebesgue integrable. In general, an increasing and continuous function need not be absolutely continuous. For instance, the Cantor Lebesgue function for the Cantor set $C_1$ of measure zero is not absolutely continuous on $I$.

Since for any $x \neq y$, by (4), $\dfrac{f_{C_\gamma}(x) - f_{C_\gamma}(y)}{x-y} \leq \dfrac{1}{1-\gamma}$, if $f_{C_\gamma}$ is differentiable at $x$, then $f_{C_\gamma}'(x) \leq \dfrac{1}{1-\gamma}$. It follows that for almost all $x$ in $I$, $f_{C_\gamma}'(x) \leq \dfrac{1}{1-\gamma}$. Therefore, for any measurable subset $E$ of $I$, since $f_{C_\gamma}'$ is Lebesgue integrable,

$$\int_E f_{C_\gamma}'(x)dx \leq \frac{1}{1-\gamma} m(E). \qquad \text{------------------------------ (6)}$$

Since $f_{C_\gamma}$ is onto,

$m(f_{C_\gamma}(I)) = m(I) = m(f_{C_\gamma}(C_\gamma \cup G) = m(f_{C_\gamma}(C_\gamma) \cup f_{C_\gamma}(G)) \le m(f_{C_\gamma}(C_\gamma)) + m(f_{C_\gamma}(G))$.

Thus,

$$1 \le m(f_{C_\gamma}(C_\gamma)) + m(f_{C_\gamma}(G)) . \text{ -------------------------- (7)}$$

By Theorem 11 of *Functions of Bounded Variation and Johnson's Indicatrix*, since $f_{C_\gamma}$ is monotone increasing and absolutely continuous, i.e., by (5) above,

$\int_G f_{C_\gamma}'(x)dx = m(f_{C_\gamma}(G))$. Since $f_{C_\gamma}'(x) = 0$ for all *x* in *G*, it follows that

$$m(f_{C_\gamma}(G)) = \int_G f_{C_\gamma}'(x)dx = 0 .$$

Therefore, it follows from (7) that $m(f_{C_\gamma}(C_\gamma)) \ge 1$ and as $m(f_{C_\gamma}(C_\gamma)) \le 1$ because $f_{C_\gamma}(C_\gamma) \subseteq I$, $m(f_{C_\gamma}(C_\gamma)) = 1$. (We may of course deduce this directly since we have shown that $f_{C_\gamma}(C_\gamma) = I$.)

For any measurable subset, *E*, of $C_\gamma$,

$$m(f_{C_\gamma}(C_\gamma)) = m\left(f_{C_\gamma}\left(E \cup \left(C_\gamma - E\right)\right)\right) \le m(f_{C_\gamma}(E)) + m(f_{C_\gamma}(C_\gamma - E)) . \text{ ----------- (8)}$$

But by (5) and (6), $m(f_{C_\gamma}(C_\gamma - E)) = \int_{C_\gamma - E} f_{C_\gamma}'(x)dx \le \frac{1}{1-\gamma} m(C_\gamma - E)$. Therefore, it follows from (8) that $1 \le m(f_{C_\gamma}(E)) + \frac{1}{1-\gamma} m(C_\gamma - E) = m(f_{C_\gamma}(E)) + \frac{1}{1-\gamma}\left(m(C_\gamma) - m(E)\right)$.

But $m(C_\gamma) = 1-\gamma$ and so we have, $1 \le m(f_{C_\gamma}(E)) + 1 - \frac{m(E)}{1-\gamma}$. It follows that

$$m(f_{C_\gamma}(E)) \ge \frac{1}{1-\gamma} m(E) .$$

Hence, by (5),

$$\int_E f_{C_\gamma}'(x)dx = m(f_{C_\gamma}(E)) \ge \frac{1}{1-\gamma} m(E)$$

and together with (6), we conclude that for any measurable subset *E* of $C_\gamma$,

$$\int_E f_{C_\gamma}'(x)dx = \frac{1}{1-\gamma} m(E) . \text{ ---------------------------- (9)}$$

Now let $D = \left\{x \in C_\gamma : f_{C_\gamma} \text{ is differentiable at } x\right\} \subseteq C_\gamma$. It follows then from (9) that

$$\int_D f_{C_\gamma}{}'(x)dx = \frac{1}{1-\gamma} m(D).$$

Hence, $\int_D \left( \frac{1}{1-\gamma} - f_{C_\gamma}{}'(x) \right) dx = 0$. Since $f_{C_\gamma}{}'(x) \le \frac{1}{1-\gamma}$ for all $x$ in $D$, we then have $\frac{1}{1-\gamma} - f_{C_\gamma}{}'(x) \ge 0$. It follows that $\frac{1}{1-\gamma} - f_{C_\gamma}{}'(x) = 0$ almost everywhere on $D$. Since $f_{C_\gamma}$ is differentiable almost everywhere with $f_{C_\gamma}$ differentiable on $(C_\gamma)^c$ and on $D$, $f_{C_\gamma}$ is not differentiable on $C_\gamma - D$ with $m(C_\gamma - D) = 0$. It follows that $f_{C_\gamma}{}'(x) = \frac{1}{1-\gamma}$ almost everywhere on $C_\gamma$.

This completes the proof of the proposition.

**Consequence of Absolute Continuity**

In the case of the Cantor Lebesgue function, $f_{C_\gamma}$, for the fat Cantor set, instead of using the polygonal approximations of $f_{C_\gamma}$, we may use the usual arc length formula for the graph of $f_{C_\gamma}$ since $f_{C_\gamma}$ is absolutely continuous.

By Theorem 9 of *Arc Length, Functions of Bounded Variation and Total Variation*, the arc length of the graph of *f* is given by,

$$\int_I \sqrt{1+\left(f_{C_\gamma}{}'(x)\right)^2}\,dx = \int_G \sqrt{1+\left(f_{C_\gamma}{}'(x)\right)^2}\,dx + \int_{C_\gamma} \sqrt{1+\left(f_{C_\gamma}{}'(x)\right)^2}\,dx$$

$$= \int_G 1dx + \int_{C_\gamma} \sqrt{1+\left(\frac{1}{1-\gamma}\right)^2}\,dx = m(G) + \frac{1}{1-\gamma}\sqrt{1+(1-\gamma)^2}\,m(C_\gamma)$$

$$= \gamma + \frac{1}{1-\gamma}\sqrt{1+(1-\gamma)^2}\cdot(1-\gamma) = \gamma + \sqrt{1+(1-\gamma)^2} \quad .$$

Moreover, for any increasing and continuous function, $f:[0,1] \to \mathbb{R}$, the arc length of the graph of *f* is given by Theorem 9 in *Arc Length, Functions of Bounded Variation and Total Variation* as

$$\int_{[0,1]} \sqrt{1+\left(f'(x)\right)^2}\,dx + T_h[0,1],$$

where $h(x)=f(x)-\int_0^x f'(x)dx$ is the singular part of *f* in the Lebesgue decomposition of *f* and $T_h[0,1]$ is the total variation of *h* on [0,1].

Therefore, for the ternary Cantor Lebesgue function, $f_{C_1}$, which is not absolutely continuous, the arc length of its graph is given by

$$\int_{[0,1]}\sqrt{1+\left(\left(f_{C_1}\right)'(x)\right)^2}\,dx+T_{f_{C_1}}[0,1],$$

since $f_{C_1}$ is singular.

Plainly, $T_{f_{C_1}}[0,1]=1$ and so its arc length is given by

$$\int_{[0,1]}\sqrt{1+\left(\left(f_{C_1}\right)'(x)\right)^2}\,dx+1=\int_{[0,1]}1dx+1=2\,.$$

## Related Cantor function

Using the construction of the Cantor set $C_\gamma$, we can similarly define a function, *g,* mapping [0, 1] onto [0, 1] such that *g* maps $C_\gamma$, $0<\gamma<1$, on to $C_1$, the usual ternary Cantor set of measure zero.

Remember that $G=[0,1]-C_\gamma=\bigcup_{k=1}^{\infty}G(k)=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}}I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$ for $n\geq 1$, $G(1)=U(1)=I(\frac{1}{2})$, $G(n+1)=G(n)\cup U(n+1)$ so that $G(n)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\leq m\leq 2^n-1\right\}$.

Note that the length of the open interval, $I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, is equal to $\frac{\gamma}{3^n}$. Hence,

$$G=\left\{I(r):r\text{ a dyadic rational, }r=\sum_{k=1}^{n}\frac{b_k}{2^k},\text{where }b_k=0\text{ or }1,\ 1\leq k<n, b_n=1,\ n\geq 1\right\}.$$

Similarly, in the construction of the ternary Cantor set, $C_1$, we have the complement of $C_1$ is given by $H=[0,1]-C_1=\bigcup_{k=1}^{\infty}H(k)=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}}J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$ for $n\geq 1$, $H(1)=V(1)=J(\frac{1}{2})$, $H(n+1)=H(n)\cup V(n+1)$ so that $H(n)=\bigcup\left\{J\left(\frac{m}{2^n}\right):1\leq m\leq 2^n-1\right\}$.

The length of the open interval, $J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, is equal to $\frac{1}{3^n}$.

Hence, $H=\left\{J(r): r \text{ a dyadic rational, } r=\sum_{k=1}^{n}\frac{b_k}{2^k}, \text{where } b_k=0 \text{ or } 1, 1\le k<n, b_n=1, n\ge 1\right\}$.

We define the function $g$ on the open intervals $I(r)$ by mapping $I(r)$ onto $J(r)$ linearly.

If $I(r)=(a(r),b(r))$ and $J(r)=(c(r),d(r))$, then for $x$ in $I(r)=(a(r),b(r))$ define

$$g(x)=\frac{d(r)-c(r)}{b(r)-a(r)}(x-a(r))+c(r)=\frac{1}{\gamma}(x-a(r))+c(r) .$$

Note that $\frac{d(r)-c(r)}{b(r)-a(r)}=\frac{\textit{length} \text{ of } J(r)}{\textit{length} \text{ of } I(r)}=\frac{1}{\gamma}$ because for $r=\sum_{k=1}^{n}\frac{b_k}{2^k}$, where $b_n=1$, the length of $J(r)$ is $\frac{1}{3^n}$ and the length of $I(r)$ is $\frac{\gamma}{3^n}$.

This defines $g$ on $G$ and plainly, $G$ is mapped onto $H$ linearly on each open interval in $G$.

Set $g(0) = 0$.

For $x \neq 0$ and $x$ in $C_\gamma$, let

$$g(x)=\sup\{g(y): y<x \text{ and } y\in G\}=\sup\{g(y): y<x \text{ and } y\in[0,1]-C_\gamma\}.$$

This is well defined by the completeness property of $\mathbb{R}$, since the set $\{g(y): y<x \text{ and } y\in G\}$ is bounded above by 1.

**Proposition 9.** The function $g$ defined above is strictly increasing, continuous and maps the interval [0,1] bijectively onto [0, 1], mapping the fat Cantor set $C_\gamma$, $0 < \gamma < 1$, onto the ternary Cantor set $C_1$ of measure zero.

**Proof.** The proof is exactly the same as the proof of Lemma 1 in *Composition and Riemann Integrability*.

**Remark.** We may define the Cantor Lebesgue function $f_{C_\gamma}$ the same way as $g$ above.

**Proposition 10.** Let $g$ be the function defined above mapping the fat Cantor set $C_\gamma$, $0 < \gamma < 1$, onto the ternary Cantor set $C_1$ of measure zero.

(i) $g'(x)=\frac{1}{\gamma}$ for $x$ in $[0,1]-C_\gamma$.

(ii) $g'(x)=0$ almost everywhere on $C_\gamma$.

(iii) The function $g$ is absolutely continuous on [0, 1].

(iv) The arc length of the graph of g is $1-\gamma+\sqrt{1+\gamma^2}$ .

**Proof.**

**(i)** Plainly, the gradient of the function *g* on each of the open intervals of $G=[0,1]-C_\gamma$ is $\frac{1}{\gamma}$, as *g* is linear on each of the open interval of *G*. Hence, $g'(x)=\frac{1}{\gamma}$ for *x* in $[0,1]-C_\gamma=G$.

**(ii)** Since $m(g(C_\gamma))=m(C_1)=0$, by Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, $g'(x)=0$ almost everywhere on $C_\gamma$.

**(iii)** Let $P=\{x\in[0,1]:g'(x)=+\infty\}$. Since *g* is differentiable on $[0,1]-C_\gamma$, $P\subseteq C_\gamma$. Therefore, $m(g(P))=m\left(g\left(\{x\in[0,1]:g'(x)=+\infty\}\right)\right)\le m(g(C_\gamma))=0$ implies that $m(g(P))=0$. Therefore, by Theorem 12 part (a) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, since *g* is strictly increasing and continuous on [0, 1], *g* is absolutely continuous on [0,1].

We may prove this slightly differently as follows.

Let $E=\{x\in[0,1]:g \text{ is differentiable at } x\}$. Then $G\subseteq E$ so that $I-E\subseteq I-G=C_\gamma$. Therefore, $m(g(I-E))=0$. Since *g* is continuous, by Lemma 4 of *When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous? (The answer and application to generalized change of variable for Lebesgue integral)*, *g* is an *N* function. Therefore, since *g* is differentiable almost everywhere and $g'$ is Lebesgue integrable, by Theorem 5 of the same article cited above, *g* is absolutely continuous on [0, 1]. (We may invoke Banach-Zarecki Theorem, Theorem 8 of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem,* since *g* is a continuous function of bounded variation and is an *N* function, to conclude that *g* is absolutely continuous.)

A theorem of Sak, Theorem 1 of *When is a continuous function on a closed and bounded interval be of bounded variation, absolutely continuous?* (*The answer and application to generalized change of variable for Lebesgue integral*), implies that *g* is absolutely continuous. This is because *g* is a continuous function of bounded variation and $g'$ is Lebesgue integrable since *g* is increasing, *g* is differentiable on $[0,1]-C_\gamma=G$ and $m\left(g(C_\gamma)\right)=0$. See also Theorem 16 of "*Absolutely Continuous Function on Arbitrary Domain and Function of Bounded Variation*". Since *g* is continuous, increasing, differentiable on $[0,1]-C_\gamma$ and $m\left(g(\{x\in[0,1]:g'(x)=+\infty\})\right)=0$ as $m\left(g(C_\gamma)\right)=0$, the condition of the theorem is met. More precisely, the theorem states that:

A continuous function $g$ of bounded variation is absolutely continuous if, and only if, $m\big(g(\{x : g'(x) = \pm\infty\})\big) = 0$.

We shall use any one of the above result to determine if any one of our Cantor function is absolutely continuous.

**(iv)** Since $g$ is absolutely continuous on [0, 1], the arc length of the graph of $g$ is given by

$$\int_I \sqrt{1+\left(g'(x)\right)^2}\,dx = \int_G \sqrt{1+\left(g'(x)\right)^2}\,dx + \int_{C_\gamma} \sqrt{1+\left(g'(x)\right)^2}\,dx$$

$$= \int_G \sqrt{1+\left(\frac{1}{\gamma}\right)^2}\,dx + \int_{C_\gamma} \sqrt{1+0}\,dx,$$

since $g'(x) = \dfrac{1}{\gamma}$ on $G$ and $g'(x) = 0$ almost everywhere on $C_\gamma$,

$$= \frac{\sqrt{1+\gamma^2}}{\gamma} m(G) + m(C_\gamma)$$

$$= \sqrt{1+\gamma^2} + 1 - \gamma\,.$$

(Of course, we can always take the limit of the arc lengths of the graph of the polygonal approximations of $g$.)

**Remark.**

The inverse function of $g$, $g^{-1}$, is strictly increasing and continuous. But $g^{-1}$ is not absolutely continuous, since $m(g^{-1}(C_1)) = m(C_\gamma) = 1-\gamma > 0$ and so cannot be an $N$ function and that a necessary condition for a function to be absolutely continuous is that it must be an $N$ function, i.e., it must map sets of measure 0 to sets of measure zero.

If $h$ is the Cantor function for the ternary Cantor set, $C_1$, of measure 0, then for the same reason as above, $h$ is not absolutely continuous because $h$ maps $C_1$ on to $I$, which is not of measure 0. But the composite $h \circ g$ is the generalised Cantor Lebesgue function for $C_\gamma$, where $0 < \gamma < 1$, and is absolutely continuous on [0, 1].

## Integral of Cantor function

Recall the polygonal approximation, $f_n : I \to I$, as defined in the proof of Proposition 6, of the generalized Cantor function, $f_{C_\gamma} : I \to I$, converges uniformly to $f$. Therefore, as each $f_n$ is integrable, $\int_I f_n(x)dx \to \int_I f_{C_\gamma}(x)dx$.

We now describe, the integral of each $f_n : I \to I$.

Recall that for each integer $k \geq 1$, $U(k) = \left\{ I\left( \frac{i}{2^{k-1}} + \frac{1}{2^k} \right) : 0 \leq i \leq 2^{k-1} - 1 \right\}$. Since the length of each open interval in $U(k)$ is $\frac{\gamma}{3^k}$, for $1 \leq k \leq n$, the integral,

$$\int_{U(k)} f_n(x)dx = \frac{\gamma}{3^k}\left\{ \left( \frac{1}{2^k} + \frac{0}{2^{k-1}} \right) + \left( \frac{1}{2^k} + \frac{2}{2^{k-1}} \right) + \cdots + \left( \frac{1}{2^k} + \frac{2^{k-1}-1}{2^{k-1}} \right) \right\}$$

$$= \frac{\gamma}{3^k} \cdot \frac{1}{2^k}\left\{ \left( \frac{1}{2^k} \right) + \left( \frac{1+2}{2^k} \right) + \cdots + \left( \frac{1+2(2^{k-1}-1)}{2^k} \right) \right\} = \frac{\gamma}{3^k} \cdot \frac{1}{2^k} \sum_{i=0}^{2^{k-1}-1} (2i+1)$$

$$= \frac{\gamma}{3^k} \cdot \frac{1}{2^k} \cdot \frac{2^{k-1}}{2}(1 + 2(2^{k-1}-1) + 1) = \frac{\gamma}{4} \cdot \left( \frac{2}{3} \right)^k .$$

As $G(n) = \bigcup_{k=1}^{n} U(k)$ is a disjoint union, $\int_{G(n)} f_n(x)dx = \frac{\gamma}{4} \sum_{k=1}^{n} \left( \frac{2}{3} \right)^k = \frac{\gamma}{2}\left( 1 - \left( \frac{2}{3} \right)^n \right)$. In particular, as $f_{C_\gamma}(x) = f_n(x)$ for $x$ in $U(n)$, $\int_{U(n)} f_{C_\gamma}(x)dx = \frac{\gamma}{4} \cdot \left( \frac{2}{3} \right)^n$ and so

$$\int_G f_{C_\gamma}(x)dx = \sum_{n=1}^{\infty} \int_{U(n)} f_{C_\gamma}(x)dx = \frac{\gamma}{4} \sum_{n=1}^{\infty} \left( \frac{2}{3} \right)^n = \frac{\gamma}{2}.$$

Now, $I - G(n) = \bigcup_{k=1}^{2^n} J\left( n, \frac{k-1}{2^n} \right)$, a disjoint union of $2^n$ closed interval of the same length, $\ell_n = \frac{1}{2^n}\left( 1 - \gamma\left( 1 - \left(\tfrac{2}{3}\right)^n \right) \right)$. Here we have indexed these closed intervals by their minimum values on each closed interval, that is, $J\left( n, \frac{i}{2^n} \right)$, $0 \leq i \leq 2^n - 1$. Therefore, $\int_{I-G(n)} f_n(x)dx$, is given by $2^n$ triangular parts of equal area given by, $2^n \cdot \frac{1}{2}\ell_n \cdot \frac{1}{2^n} = \frac{1}{2}\ell_n$ plus the rectangular parts below these triangles, given by

$$\ell_n \times \sum_{i=1}^{2^n-1} \frac{i}{2^n} = \ell_n \cdot \frac{1}{2^n} \cdot \frac{2^n - 1}{2} \cdot 2^n = \frac{1}{2} 2^n \ell_n - \frac{1}{2}\ell_n = \frac{1}{2}\left( 1 - \gamma\left( 1 - \left(\tfrac{2}{3}\right)^n \right) \right) - \frac{1}{2}\ell_n .$$

It follows that $\int_{I-G(n)} f_n(x)dx = \frac{1}{2}\ell_n + \frac{1}{2}\left( 1 - \gamma\left( 1 - \left(\tfrac{2}{3}\right)^n \right) \right) - \frac{1}{2}\ell_n = \frac{1}{2}\left( 1 - \gamma\left( 1 - \left(\tfrac{2}{3}\right)^n \right) \right)$. Hence,

$\int_I f_n(x)dx = \frac{\gamma}{2}\left(1-\left(\frac{2}{3}\right)^n\right)+\frac{1}{2}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)=\frac{1}{2}$. It follows that $\int_I f_{C_\gamma}(x)dx = \frac{1}{2}$. Hence,

$\int_{C_\gamma} f_{C_\gamma}(x)dx = \int_{[0,1]} f_{C_\gamma}(x)dx - \int_G f_{C_\gamma}(x)dx = \frac{1}{2}-\frac{\gamma}{2}=\frac{1-\gamma}{2}$.

Note that this is also true for $\gamma=1$.

Hence, we have:

**Proposition 11**. The integral of the generalized Cantor function, $f_{C_\gamma}$, including the usual ternary Cantor function, when $\gamma=1$, is ½. $\int_{I-C_\gamma} f_{C_\gamma}(x)dx = \frac{\gamma}{2}$ and $\int_{C_\gamma} f_{C_\gamma}(x)dx = \frac{1-\gamma}{2}$.

We can also prove this by proving that for the Cantor function, $f_{C_\gamma}$ (with $0 < \gamma \leq 1$), $f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=1$ for all $x$ in $I$. Indeed $\int_I f_{C_\gamma}(x)dx + \int_I f_{C_\gamma}(1-x)dx = 1$. But

$$\int_I f_{C_\gamma}(1-x)dx = \int_0^1 f_{C_\gamma}(1-x)dx = -\int_1^0 f_{C_\gamma}(u)du \text{ , with substitution } u = 1-x \text{ ,}$$
$$= \int_0^1 f_{C_\gamma}(u)du = \int_0^1 f_{C_\gamma}(x)dx .$$

It follows that $2\int_I f_{C_\gamma}(x)dx = 1$ and so $\int_I f_{C_\gamma}(x)dx = \frac{1}{2}$.

We may prove this relation on each polygonal approximation $f_n$ and then take the limit to infinity on each side of the relation.

**Proposition 12.** The associated Cantor Lebesgue function $f_{C_\gamma}: I \to I$ for the Cantor set $C_\gamma$, $0 < \gamma \leq 1$, satisfies $f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=1$ for all $x$ in $I$.

**Proof.**

It is enough to show that the polygonal approximation, $f_n: I \to I$, of $f$, satisfies the same relation $f_n(x)+f_n(1-x)=1$ for all $x$ in $I$ and for all integer $n \geq 1$.

We first show that $f_n(x)+f_n(1-x)=1$ for all $x$ in $G(n)$.

Note that $G(n) = \bigcup\left\{ I\left(\frac{m}{2^n}\right) : 1 \leq m \leq 2^n - 1 \right\}$. Notice by the construction of $G(n)$, the open sets of $G(n)$ are symmetrical about the point $y$ = ½, with $\frac{1}{2} \in I(1/2)$. More precisely, if $h(x) = 1-x$, then $h(I(r)) = I(1-r)$.

Suppose $x \in I(r)$, $r = \dfrac{k}{2^n}, 1 \le k \le 2^n - 1$. We shall show that $h(x) = 1 - x \in I(1-r)$.

We show that $h(I(r)) = I(1-r)$ by induction on *n*.

When *n* =1, then $G(1) = U(1) = I(1/2)$, obviously $h(I(1/2)) = I(1-1/2) = I(1/2)$.

$G(2) = G(1) \cup U(2)$ and $U(2) = \left\{ I\left(\dfrac{i}{2} + \dfrac{1}{2^2}\right) : 0 \le i \le 1 \right\}$. Since $I - G(1)$ is a disjoint union of two closed interval of equal length and is symmetrical placed about *y* = ½, the open intervals, $I\left(\dfrac{1}{2^2}\right)$ and $I\left(\dfrac{3}{2^2}\right)$, are symmetrical about *y* = ½.

$$x \in I\left(\frac{1}{2^2}\right) \Leftrightarrow \ell_2 < x < \ell_2 + \frac{\gamma}{3^2} \Leftrightarrow 1 - \ell_2 - \frac{\gamma}{3^2} < 1 - x < 1 - \ell_2 \Leftrightarrow 1 - x \in I\left(1 - \frac{1}{2^2}\right).$$

Thus, $h\left(I\left(\dfrac{1}{2^2}\right)\right) = I\left(1 - \dfrac{1}{2^2}\right)$.

Suppose $h(I(r)) = I(1-r)$ for $I(r) \in G(k) = \bigcup \left\{ I\left(\dfrac{m}{2^k}\right) : 1 \le m \le 2^k - 1 \right\}$.

Now, $G(k+1) = \bigcup \left\{ I\left(\dfrac{m}{2^k}\right) : 1 \le m \le 2^k - 1 \right\} \cup \left\{ I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) : 0 \le m \le 2^k - 1 \right\}$. Recall that $U(k+1) = \left\{ I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) : 0 \le m \le 2^k - 1 \right\}$. We let $I(0) = \{0\}$. Note that $I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) = I\left(\dfrac{2m+1}{2^{k+1}}\right)$ and so $I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) = I\left(\dfrac{2m+1}{2^{k+1}}\right)$ is between $I\left(\dfrac{2m}{2^{k+1}}\right)$ and $I\left(\dfrac{2m+2}{2^{k+1}}\right)$. The map *h* maps $I\left(\dfrac{2m}{2^{k+1}}\right)$ to $I\left(1 - \dfrac{2m}{2^{k+1}}\right) = I\left(\dfrac{2^{k+1} - 2m}{2^{k+1}}\right)$ and maps $I\left(\dfrac{2m+2}{2^{k+1}}\right)$ to $I\left(1 - \dfrac{2m+2}{2^{k+1}}\right) = I\left(\dfrac{2^{k+1} - 2m - 2}{2^{k+1}}\right)$. $I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) = I\left(\dfrac{2m+1}{2^{k+1}}\right)$ is placed in the middle interval of the closed interval between $I\left(\dfrac{2m}{2^{k+1}}\right)$ and $I\left(\dfrac{2m+2}{2^{k+1}}\right)$ of length $\dfrac{\gamma}{3^{k+1}}$ and so *h* must map $I\left(\dfrac{m}{2^k} + \dfrac{1}{2^{k+1}}\right) = I\left(\dfrac{2m+1}{2^{k+1}}\right)$ to the middle interval of the closed interval between $I\left(1 - \dfrac{2m+2}{2^{k+1}}\right) = I\left(\dfrac{2^{k+1} - 2m - 2}{2^{k+1}}\right)$ and $I\left(1 - \dfrac{2m}{2^{k+1}}\right) = I\left(\dfrac{2^{k+1} - 2m}{2^{k+1}}\right)$, which is of course $I\left(\dfrac{2^{k+1} - 2m - 1}{2^{k+1}}\right) = I\left(1 - \dfrac{2m+1}{2^{k+1}}\right)$. This proves that $h(I(r)) = I(1-r)$ for

$I(r) \in U(k+1)$. Hence, $h(I(r)) = I(1-r)$ for $I(r) \in G(k+1) = \bigcup \left\{ I\left(\frac{m}{2^{k+1}}\right) : 1 \le m \le 2^{k+1} - 1 \right\}$ and for any $k \ge 1$. It follows that $h(I(r)) = I(1-r)$ for any $I(r)$ in $G$. Hence, if $x$ is in $I(r) \subseteq G(n)$, then $1 - x \in I(1-r)$. Consequently, $f_n(x) + f_n(1-x) = r + 1 - r = 1$. Plainly this applies also to the end points of the open intervals of $G(n)$. It follows that $f_n(x) + f_n(1-x) = 1$ for all $x \in \overline{G(n)}$ .

Suppose now $x \notin \overline{G(n)}$ . Then $x$ is in one of the interior of the closed intervals in $I - G(n)$. Each of these closed intervals are of the same length $\ell_n$. We can index these closed intervals by the max value of $f_n$ on it, i.e., $I - G(n) = \left\{ J\left(n, \frac{i}{2^n}\right) : 1 \le i \le 2^n \right\}$.

Suppose $x$ is in the interior of $J\left(n, \frac{k}{2^n}\right)$, $1 \le k \le 2^n$. For technicality, let $I(1) = \{1\}$ and $I(0) = \{0\}$. Then $J\left(n, \frac{k}{2^n}\right)$ is the closed interval between $I\left(\frac{k-1}{2^n}\right)$ and $I\left(\frac{k}{2^n}\right)$ for $1 \le k \le 2^n$. Since $h\left(I\left(\frac{k-1}{2^n}\right)\right) = I\left(1 - \frac{k-1}{2^n}\right)$ and $h\left(I\left(\frac{k}{2^n}\right)\right) = I\left(1 - \frac{k}{2^n}\right)$, $h\left(J\left(n, \frac{k}{2^n}\right)\right)$ is the closed interval between $I\left(1 - \frac{k}{2^n}\right)$ and $I\left(1 - \frac{k-1}{2^n}\right)$. Let $I\left(\frac{k-1}{2^n}\right) = \left(a_{k-1}, b_{k-1}\right)$ and $I\left(\frac{k}{2^n}\right) = \left(a_k, b_k\right)$. Then $I\left(1 - \frac{k}{2^n}\right) = (1 - b_k, 1 - a_k)$ and $I\left(1 - \frac{k-1}{2^n}\right) = (1 - b_{k-1}, 1 - a_{k-1})$.

Note that $a_k - b_{k-1} = \ell_n$ , $b_{k-1} < x < a_k$ and $1 - a_k < 1 - x < 1 - b_{k-1}$

Now $f_n(x) = \frac{k-1}{2^n} + \left(x - b_{k-1}\right)\frac{1}{2^n \ell_n}$ and

$$f_n(1-x) = 1 - \frac{k}{2^n} + \left(1 - x - \left(1 - a_k\right)\right)\frac{1}{2^n \ell_n} = 1 - \frac{k}{2^n} + \left(a_k - x\right)\frac{1}{2^n \ell_n}.$$

Therefore,

$$\begin{aligned} f_n(x) + f_n(1-x) &= \frac{k-1}{2^n} + \left(x - b_{k-1}\right)\frac{1}{2^n \ell_n} + 1 - \frac{k}{2^n} + \left(a_k - x\right)\frac{1}{2^n \ell_n} \\ &= 1 - \frac{1}{2^n} + \left(a_k - b_{k-1}\right)\frac{1}{2^n \ell_n} = 1 - \frac{1}{2^n} + \ell_n \frac{1}{2^n \ell_n} = 1. \end{aligned}$$

Note that in the above proceeding, if $k$ =1, then $I\left(\frac{k-1}{2^n}\right)=I(0)=\{0\}=\{b_0\}$ and $I\left(1-\frac{k-1}{2^n}\right)=I(1)=\{1\}=\{1-b_0\}$. If $k=2^n$, then $I\left(\frac{k}{2^n}\right)=I(1)=\{1\}=\{a_k\}$ and $I\left(1-\frac{k}{2^n}\right)=I(0)=\{0\}=\{1-a_k\}$.

This proves that $f_n(x)+f_n(1-x)=1$ for all $x$ in $I$ and for all integer $n \geq 1$. Therefore,

$$f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=\lim_{n\to\infty} f_n(x)+\lim_{n\to\infty} f_n(1-x)=\lim_{n\to\infty}\left(f_n(x)+f_n(1-x)\right)=1 \text{ for all } x \text{ in } I.$$

*Alternatively*, we may proceed as follows.

We have already shown that for all $x$ in $G$, $f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=1$. For $x\in I-G=C_\gamma$, there exists a sequence $(a_n)$ in $G$ such that $a_n \to x$. Since $f_{C_\gamma}$ is continuous, $f_{C_\gamma}(a_n)\to f_{C_\gamma}(x)$ and $f_{C_\gamma}(1-a_n)\to f_{C_\gamma}(1-x)$. Since $a_n \in G$, $f_{C_\gamma}(a_n)+f_{C_\gamma}(1-a_n)=1$. It follows that,

$$f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=\lim_{n\to\infty} f_{C_\gamma}(a_n)+\lim_{n\to\infty} f_{C_\gamma}(1-a_n)=\lim_{n\to\infty}\left(f_{C_\gamma}(a_n)+f_{C_\gamma}(1-a_n)\right)=1.$$

Therefore, $f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=1$ for all $x\in I-G=C_\gamma$. Hence, $f_{C_\gamma}(x)+f_{C_\gamma}(1-x)=1$ for all $x$ in $I$.

The function, $g$, as defined just prior to Proposition 9 and referred to in Proposition 9, also satisfies the relation $g(x)+g(1-x)=1$ for all $x$ in $I$.

By Proposition 9, $g$ is strictly increasing, continuous and maps the interval [0,1] bijectively onto [0, 1], mapping the fat Cantor set $C_\gamma$ ( $0 < \gamma < 1$) onto the ternary Cantor set $C_1$ of measure zero.

Let $G=[0,1]-C_\gamma=\bigcup_{k=1}^{\infty} G(k)=\bigcup_{k=1}^{\infty} U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}} I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$,

$G(n)=\bigcup_{k=1}^{n} U(k)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\leq m\leq 2^n-1\right\}$ and $G(n+1)=G(n)\cup U(n+1)$.

Let $H=[0,1]-C_1=\bigcup_{k=1}^{\infty} H(k)=\bigcup_{k=1}^{\infty} V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}} J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$ for $n \geq 1$, are the middle third open intervals in the construction of $C_1$,

$H(n)=\bigcup_{k=1}^{n} V(k)=\bigcup\left\{J\left(\frac{m}{2^n}\right):1\leq m\leq 2^n-1\right\}$ and $H(n+1)=H(n)\cup U(n+1)$.

The function $g$ maps each $I(r)$, linearly, strictly increasingly and bijectively onto $J(r)$.

For each integer $n \geq 1$, let $g_n : I \to I$ be the polygonal approximation of $g$ such that $g_n(x) = g(x)$ for $x$ in each $I\left(\frac{m}{2^n}\right)$ in $G(n)$. As before we can index the closed intervals in $I - G(n)$ all of equal lengths, $\ell_n = \frac{1}{2^n}\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)$, by $I - G(n) = \left\{S\left(n, \frac{i}{2^n}\right) : 1 \leq i \leq 2^n\right\}$. Similarly, we index the closed intervals in $I - H(n)$ all of equal lengths, $\lambda_n = \frac{1}{2^n}\left(1-\left(1-\left(\frac{2}{3}\right)^n\right)\right)$, by $I - H(n) = \left\{T\left(n, \frac{i}{2^n}\right) : 1 \leq i \leq 2^n\right\}$. Thus, if $x \in I(r) = (a(r), b(r))$ and $J(r) = (c(r), d(r))$, $g_n(x) = g(x) = c(r) + \frac{d(r)-c(r)}{b(r)-a(r)}(x - a(r))$. Now, $S\left(n, \frac{k}{2^n}\right)$, is the closed interval between $I\left(\frac{k-1}{2^n}\right)$ and $I\left(\frac{k}{2^n}\right)$. $T\left(n, \frac{k}{2^n}\right)$, is the closed interval between $J\left(\frac{k-1}{2^n}\right)$ and $J\left(\frac{k}{2^n}\right)$.

We use the convention: $I(1) = J(1) = \{1\}$ and $I(0) = J(0) = \{0\}$. $g_n$ maps $S\left(n, \frac{k}{2^n}\right)$ linearly and bijectively onto $T\left(n, \frac{k}{2^n}\right)$ as follows.

Let $I\left(\frac{k-1}{2^n}\right) = \left(a(\tfrac{k-1}{2^n}), b(\tfrac{k-1}{2^n})\right)$, $I\left(\frac{k}{2^n}\right) = \left(a(\tfrac{k}{2^n}), b(\tfrac{k}{2^n})\right)$, $J\left(\frac{k-1}{2^n}\right) = \left(c(\tfrac{k-1}{2^n}), d(\tfrac{k-1}{2^n})\right)$, $J\left(\frac{k}{2^n}\right) = \left(c(\tfrac{k}{2^n}), d(\tfrac{k}{2^n})\right)$. Then for $x \in S\left(n, \frac{k}{2^n}\right)$,

$$g_n(x) = d(\tfrac{k-1}{2^n}) + \frac{c(\tfrac{k}{2^n}) - d(\tfrac{k-1}{2^n})}{a(\tfrac{k}{2^n}) - b(\tfrac{k-1}{2^n})}(x - b(\tfrac{k-1}{2^n})) = d(\tfrac{k-1}{2^n}) + \frac{\lambda_n}{\ell_n}(x - b(\tfrac{k-1}{2^n}))$$

$$= d(\tfrac{k-1}{2^n}) + \frac{\left(1-\left(1-\left(\frac{2}{3}\right)^n\right)\right)}{\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)}(x - b(\tfrac{k-1}{2^n})) = d(\tfrac{k-1}{2^n}) + \frac{\left(\frac{2}{3}\right)^n}{\left(1-\gamma\left(1-\left(\frac{2}{3}\right)^n\right)\right)}(x - b(\tfrac{k-1}{2^n})).$$

Let $h(x) = 1 - x$. We shall show that $g_n(x) + g_n(1-x) = 1$ for all $x$ in $G(n)$. We shall do this by induction. For $n = 1$, $G(1) = U(1) = I(1/2)$, obviously $h(I(1/2)) = I(1-1/2) = I(1/2)$. Now $g(I(1/2)) = g_n(I(1/2)) = J(1/2)$. Let $I(1/2) = \left(a(\tfrac{1}{2}), b(\tfrac{1}{2})\right)$ and $J(1/2) = \left(c(\tfrac{1}{2}), d(\tfrac{1}{2})\right)$. Then for $x$ in $G(1) = I(1/2) = \left(a(\tfrac{1}{2}), b(\tfrac{1}{2})\right)$,

$$g_1(x) = c(\tfrac{1}{2}) + \frac{d(\tfrac{1}{2}) - c(\tfrac{1}{2})}{b(\tfrac{1}{2}) - a(\tfrac{1}{2})}(x - a(\tfrac{1}{2})) = c(\tfrac{1}{2}) + \frac{\frac{1}{3}}{\gamma\frac{1}{3}}(x - a(\tfrac{1}{2})) = c(\tfrac{1}{2}) + \frac{1}{\gamma}(x - a(\tfrac{1}{2}))$$

and $g_1(1-x) = c(\tfrac{1}{2}) + \frac{d(\tfrac{1}{2}) - c(\tfrac{1}{2})}{b(\tfrac{1}{2}) - a(\tfrac{1}{2})}(1 - x - a(\tfrac{1}{2})) = c(\tfrac{1}{2}) + \frac{1}{\gamma}(1 - x - a(\tfrac{1}{2}))$. But $1 - a(\tfrac{1}{2}) = b(\tfrac{1}{2})$.

It follows that

$$g_1(x)+g_1(1-x)=c(\tfrac{1}{2})+\frac{1}{\gamma}(x-a(\tfrac{1}{2}))+c(\tfrac{1}{2})+\frac{1}{\gamma}(b(\tfrac{1}{2})-x)=2c(\tfrac{1}{2})+\frac{1}{\gamma}(b(\tfrac{1}{2})-a(\tfrac{1}{2}))$$
$$=2c(\tfrac{1}{2})+d(\tfrac{1}{2})-c(\tfrac{1}{2})=c(\tfrac{1}{2})+d(\tfrac{1}{2})=1.$$

So, the relation $g_n(x)+g_n(1-x)=1$ holds for all $x$ in $G(1)$.

Now $G(2)=G(1)\cup U(2)=G(1)\cup I\left(\frac{1}{2^2}\right)\cup I\left(\frac{3}{2^2}\right)$ and $h\left(I\left(\frac{1}{2^2}\right)\right)=I\left(\frac{3}{2^2}\right)$. We shall show that $g_n(x)+g_n(1-x)=1$ holds for $x\in U(2)=\left\{I(\tfrac{i}{2}+\tfrac{1}{2^2}):0\le i\le 1\right\}$.

Let $I\left(\frac{1}{2^2}\right)=(a(\tfrac{1}{2^2}),b(\tfrac{1}{2^2}))$ and $J\left(\frac{1}{2^2}\right)=(c(\tfrac{1}{2^2}),d(\tfrac{1}{2^2}))$ Then $I\left(\frac{3}{2^2}\right)=(1-b(\tfrac{1}{2^2}),1-a(\tfrac{1}{2^2}))$ and $J\left(\frac{3}{2^2}\right)=(1-d(\tfrac{1}{2^2}),1-c(\tfrac{1}{2^2}))$.

If $x\in I\left(\frac{1}{2^2}\right)$, then $g_2(x)=c(\tfrac{1}{2^2})+\frac{1}{\gamma}(x-a(\tfrac{1}{2^2}))$ and $g_2(1-x)=1-d(\tfrac{1}{2^2})+\frac{1}{\gamma}\left(1-x-(1-b(\tfrac{1}{2^2}))\right)$.

Thus,

$$g_2(x)+g_2(1-x)=c(\tfrac{1}{2^2})+\frac{1}{\gamma}(x-a(\tfrac{1}{2^2}))+1-d(\tfrac{1}{2^2})+\frac{1}{\gamma}\left(1-x-(1-b(\tfrac{1}{2^2}))\right)$$
$$=1-d(\tfrac{1}{2^2})+c(\tfrac{1}{2^2})+\frac{1}{\gamma}(b(\tfrac{1}{2^2})-a(\tfrac{1}{2^2}))=1-d(\tfrac{1}{2^2})+c(\tfrac{1}{2^2})+d(\tfrac{1}{2^2})-c(\tfrac{1}{2^2})=1.$$

Similarly, we show that if $x\in I\left(\frac{3}{2^2}\right)$, then $g_2(x)+g_2(1-x)=1$.

We assume that $g_k(x)+g_k(1-x)=1$ for integer $k\ge 2$ and for all $x$ in $G(k)$. Now $G(k+1)=G(k)\cup U(k+1)$. We show that $g_{k+1}(x)+g_{k+1}(1-x)=1$ for $x$ in $U(k+1)$.

Note that if $I(r)$ is one of the open interval in $U(k+1)$, then $I(1-r)$ is also one of the open interval in $U(k+1)$ and moreover $h(I(r))=I(1-r)$. Note that $g$ maps any open interval, $I(r)$, in $G$ to the corresponding open interval $J(r)$ in $H$.

Let $I(r)=(a(r),b(r))$ and $J(r)=(c(r),d(r))$ Then $I(1-r)=(1-b(r),1-a(r))$ and $J(1-r)=(1-d(r),1-c(r))$.

If $x\in I(r)$, then $g_{k+1}(x)=g(x)=c(r)+\frac{1}{\gamma}(x-a(r))$ and

$$g_{k+1}(1-x)=1-d(r)+\frac{1}{\gamma}\left(1-x-(1-b(r))\right).$$

It follows that

$$g_{k+1}(x)+g_{k+1}(1-x)=c(r)+\frac{1}{\gamma}(x-a(r))+1-d(r)+\frac{1}{\gamma}\left(1-x-(1-b(r))\right)$$

$$= 1 - d(r) + c(r) + \frac{1}{\gamma}(b(r) - a(r)) = 1 - d(r) + c(r) + d(r) - c(r) = 1 .$$

This proves that $g_n(x) + g_n(1-x) = g(x) + g(1-x) = 1$ for all $x$ in $G(n)$ and for all integer $n \geq 1$. Therefore, $g(x) + g(1-x) = 1$ for all $x$ in $G$.

(Using the same technique as for the case of $f_{C_\gamma}$, the Cantor Lebesgue function, we can show that $g_n(x) + g_n(1-x) = 1$, for all $x$ in $C_\gamma$. We then prove the relation by taking limit of both sides of the relation. For this we need to use the convergence of $g_n$ to $g$. Note that $|g_n(x) - g_{n+1}(x)| \leq \frac{1}{3^n}$ for all $x$ in $I$ so that $(g_n(x))$ does converge uniformly to $g$ on $I$.)

Take any $x$ in $I - G = C_\gamma$. Since $G$ is dense in $I$, there exists a sequence $(a_n)$ in $G$ such that $a_n \to x$. As $g$ is continuous, $g(a_n) \to g(x)$ and $g(1-a_n) \to g(1-x)$. Note that $g(a_n) + g(1-a_n) = 1$ for all integer $n \geq 1$.
It follows that $g(x) + g(1-x) = \lim_{n\to\infty} g(a_n) + \lim_{n\to\infty} g(1-a_n) = \lim_{n\to\infty}\left(g(a_n) + g(1-a_n)\right) = 1$.
Hence, $g(x) + g(1-x) = 1$ for all $x$ in $I$.

In summary, we have:

**Proposition 13.** The function $g$, as defined just prior to Proposition 9 that maps the interval [0,1] bijectively onto [0, 1] and the fat Cantor set $C_\gamma$ ( $0 < \gamma < 1$) onto the ternary Cantor set $C_1$ of measure zero satisfies the relation $g(x) + g(1-x) = 1$ for all $x$ in $I$. Consequently, $\int_0^1 g(x)dx = \int_I g(x)dx = \frac{1}{2}$ .

**Remark.**

Let $f_{C_\gamma}$ be the Lebesgue Cantor function for $C_\gamma$, $0 < \gamma < 1$ and $f_{C_1}$ the Lebesgue Cantor function for $C_1$. Then $f_{C_1} \circ g = f_{C_\gamma}$. By Proposition 10, $g$ is absolutely continuous on $I$. Then by Theorem 8 of *Change of Variables Theorems*,

$$\int_0^1 f_{C_1}(x)dx = \int_0^1 f_{C_1} \circ g(x) g'(x)dx = \int_0^1 f_{C_\gamma}(x) g'(x)dx$$

$$= \int_{C_\gamma} f_{C_\gamma}(x) g'(x)dx + \int_{I-C_\gamma} f_{C_\gamma}(x) g'(x)dx = \int_{C_\gamma} f_{C_\gamma}(x) g'(x)dx + \int_G f_{C_\gamma}(x) g'(x)dx$$

$$= 0 + \int_G f_{C_\gamma}(x) \cdot \frac{1}{\gamma} dx = \frac{1}{\gamma} \int_G f_{C_\gamma}(x)dx ,$$

since $g'(x)=0$ almost everywhere on $C_\gamma$ and $g'(x)=\dfrac{1}{\gamma}$ on $G$.

Therefore, $\frac{1}{\gamma}\int_G f_{C_\gamma}(x)dx=\frac{1}{2}$ and so $\int_G f_{C_\gamma}(x)dx=\frac{\gamma}{2}$ and $\int_{C_\gamma} f_{C_\gamma}(x)dx=\frac{1}{2}-\frac{\gamma}{2}=\frac{1-\gamma}{2}$.

Thus, we have another proof of Proposition 11 for the case of fat Cantor set:

If $f_{C_\gamma}: I\to I$ is the Cantor Lebesgue function for the fat Cantor set $C_\gamma$ ( $0<\gamma<1$), then $\int_{I-C_\gamma} f_{C_\gamma}(x)dx=\frac{\gamma}{2}$ and $\int_{C_\gamma} f_{C_\gamma}(x)dx=\frac{1-\gamma}{2}$.

**Proposition 14.** Let the function, $g$, be defined just prior to Proposition 9 that maps the interval [0,1] bijectively onto [0, 1] and the fat Cantor set $C_\gamma$ ( $0<\gamma<1$) onto the ternary Cantor set $C_1$ of measure zero. Then $\int_{I-C_\gamma} g(x)dx=\frac{\gamma}{2}$ and $\int_{C_\gamma} g(x)dx=\frac{1-\gamma}{2}$.

**Proof.**

Let $G=[0,1]-C_\gamma=\bigcup_{k=1}^{\infty}G(k)=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}} I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$.

Let $H=[0,1]-C_1=\bigcup_{k=1}^{\infty}H(k)=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}} J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$ for $n\geq 1$, are the middle third open intervals in the construction of $C_1$,

$H(n)=\bigcup_{k=1}^{n}V(k)=\bigcup\left\{J\left(\frac{m}{2^n}\right):1\leq m\leq 2^n-1\right\}$ and $H(n+1)=H(n)\cup U(n+1)$.

We shall examine the open intervals in $H$ more carefully.

Let $J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)=\left(c(n,k),d(n,k)\right)$ for $1\leq k\leq 2^{n-1}$. Then $d(n,k)-c(n,k)=\frac{1}{3^n}$ and the collection $\left\{c(n,k):1\leq k\leq 2^{n-1}\right\}$ is given by $\left\{2r+\frac{1}{3^n}:r=\sum_{k=1}^{n-1}\frac{\varepsilon_k}{3^k},\varepsilon_k=0\text{ or }1\right\}$.

Therefore, since $g$ maps the intervals in $U(n)$ linearly on to the open intervals in $V(n)$, the integral of $g$ over $U(n)$ is given by

$$\begin{aligned}\int_{U(n)}g(x)dx&=\frac{\gamma}{3^n}\times\sum_{k=1}^{2^{n-1}}c(n,k)+2^{n-1}\left(\frac{1}{2}\times\frac{\gamma}{3^n}\times\frac{1}{3^n}\right)\\&=\frac{\gamma}{3^n}\times\left(2\cdot 2^{n-2}\sum_{k=1}^{n-1}\frac{1}{3^k}+2^{n-1}\cdot\frac{1}{3^n}\right)+2^{n-1}\left(\frac{1}{2}\times\frac{\gamma}{3^n}\times\frac{1}{3^n}\right)\\&=\frac{\gamma}{3^n}\times\left(2^{n-1}\sum_{k=1}^{n}\frac{1}{3^k}\right)+2^{n-2}\frac{\gamma}{3^{2n}}=2^{n-1}\frac{\gamma}{3^n}\times\left(\frac{1}{3}\cdot\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}\right)+2^{n-2}\frac{\gamma}{3^{2n}}\end{aligned}$$

$$=\frac{1}{2}2^{n-1}\frac{\gamma}{3^n}\times\left(1-\frac{1}{3^n}\right)+2^{n-2}\frac{\gamma}{3^{2n}}=\frac{1}{2}\cdot\frac{2^{n-1}}{3^n}\gamma .$$

Therefore, $\int_G g(x)dx=\sum_{n=1}^{\infty}\int_{U(n)} g(x)dx=\frac{\gamma}{2}\cdot\frac{1}{3}\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^{n-1}}=\frac{\gamma}{2}$. Consequently, $\int_{C_\gamma} g(x)dx=\frac{1-\gamma}{2}$.

We now look at similar construction of Cantor Lebesgue type function. Let $0<\gamma_1,\gamma_2\leq 1$ and $\gamma_1\neq\gamma_2$. Let $C_{\gamma_1}$ and $C_{\gamma_2}$ be the Cantor sets corresponding to $\gamma_1$ and $\gamma_2$. Let $g_{1,2}:I\to I$ be defined in exactly the same manner as *g* in Proposition 9, by first defining the functions on the disjoint open intervals in $I-C_{\gamma_1}$ linearly and bijectively onto the corresponding open intervals in $I-C_{\gamma_2}$. As $C_{\gamma_1}$ is dense in *I*, the function is then extended to the whole of *I*.

**Proposition 15.** The function, $g_{1,2}:I\to I$ , as defined above is strictly increasing and continuous and maps the Cantor set $C_{\gamma_1}$ bijectively onto $C_{\gamma_2}$, where $0<\gamma_1,\gamma_2\leq 1$ and $\gamma_1\neq\gamma_2$.

**Proof.**

Let $G=[0,1]-C_{\gamma_1}=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}}I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $I(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\leq k\leq 2^{n-1}$, are the open intervals, each of length $\frac{\gamma_1}{3^n}$, to be deleted in stage *n* in the construction of $C_{\gamma_1}$ . Let $H=[0,1]-C_{\gamma_2}=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}}J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $J(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\leq k\leq 2^{n-1}$, are the open intervals, each of length $\frac{\gamma_2}{3^n}$, to be deleted in stage *n* in the construction of $C_{\gamma_2}$. Let $G(n)==\bigcup_{k=1}^{n}U(k)$ and $H(n)=\bigcup_{k=1}^{n}V(k)$. Then $G=\bigcup_{k=1}^{\infty}G(k)$ and $H=\bigcup_{k=1}^{\infty}H(k)$. $I-G(n)=F(n,1)\cup F(n,2)\cup\cdots\cup F(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\ell_n=\frac{1}{2^n}\left(1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)\right)$ and $I-H(n)=K(n,1)\cup K(n,2)\cup\cdots\cup K(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\lambda_n=\frac{1}{2^n}\left(1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)\right)$.

Note that

$$G=\bigcup\left\{I(r): r \text{ a dyadic rational, } r=\sum_{k=1}^{n}\frac{b_k}{2^k}, \text{where } b_k=0 \text{ or } 1 \text{ for } 1\leq k<n, b_n=1,\ n\geq 1\right\}$$

and

$$H = \bigcup \left\{ J(r) : r \text{ a dyadic rational, } r = \sum_{k=1}^{n} \frac{b_k}{2^k}, \text{where } b_k = 0 \text{ or } 1 \text{ for } 1 \le k < n, b_n = 1, \, n \ge 1 \right\}.$$

The function $g_{1,2}$ is defined on the open interval $I(r)$ by mapping $I(r)$ onto $J(r)$ linearly. If $I(r) = (a(r), b(r))$ and $J(r) = (c(r), d(r))$, then for $x$ in $I(r) = (a(r), b(r))$ define

$$g(x) = \frac{d(r) - c(r)}{b(r) - a(r)}(x - a(r)) + c(r) = \frac{\gamma_2}{\gamma_1}(x - a(r)) + c(r) \ .$$

Note that $\frac{d(r)-c(r)}{b(r)-a(r)} = \frac{\textit{length} \text{ of } J(r)}{\textit{length} \text{ of } I(r)} = \frac{\gamma_2}{\gamma_1}$ since if $r = \sum_{k=1}^{n} \frac{b_k}{2^k}$, where $b_n = 1$, then the length of $J(r)$ is $\frac{\gamma_2}{3^n}$ and the length of $I(r)$ is $\frac{\gamma_1}{3^n}$.

The end points of $I(r)$ are mapped to the corresponding end points of $J(r)$.

Define $g_{1,2}(0) = 0$ and $g_{1,2}(1) = 1$.

For $x \ne 0$ and $x$ in $C_{\gamma_1}$, define

$$g_{1,2}(x) = \sup\left\{ g_{1,2}(y) : y < x \text{ and } y \in G \right\} = \sup\left\{ g_{1,2}(y) : y < x \text{ and } y \in [0,1] - C_{\gamma_1} \right\}.$$

Now we examine the indexing of $U(n)$ and $V(n)$. We note that

$$\left\{ \frac{k-1}{2^{n-1}} + \frac{1}{2^n} : 1 \le k \le 2^{n-1} \right\} = \left\{ \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2^j} + \frac{1}{2^n} : \varepsilon_j = 0 \text{ or } 1 \right\}.$$

If we let $J\left( \frac{k-1}{2^{n-1}} + \frac{1}{2^n} \right) = J\left( \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2^j} + \frac{1}{2^n} \right) = (c(r), d(r))$, where $r = \sum_{j=1}^{n-1} \frac{\varepsilon_j}{2^j} + \frac{1}{2^n}$, then

$$c(r) = \sum_{j=1}^{n-1} \left( \left( \lambda_j + \frac{\gamma_2}{3^j} \right) \varepsilon_j + \lambda_n \right).$$

Firstly, we show that $g_{1,2}$ *is strictly increasing on* $G = I - C_{\gamma_1}$.

By definition, $g_{1,2}$ is increasing on each $I(r)$ and maps $I(r)$ bijectively onto $J(r)$. At the $n$-th stage of the construction of the Cantor set we obtained
$G(n) = \bigcup_{k=1}^{n} U(k) = \bigcup \left\{ I\left( \frac{m}{2^n} \right) : 1 \le m \le 2^n - 1 \right\}$, which consists of $2^n - 1$ disjoint open intervals, that have been deleted from [0, 1]. The ordering of these intervals is in the order of the deletion starting from the left to the right according to the indexing. The natural ordering

follows a very simple rule, $I(k) < I(l)$, if and only if, there exists some $x$ in $I(k)$ such that $x < y$ for some $y$ in $I(l)$, if and only if, for any $x$ in $I(k)$, $x < y$ for all $y$ in $I(l)$. Therefore, the disjoint open sets in $G$ are ordered by its index set. Similarly, the disjoint open sets in $H = I - C_{\gamma_2}$ are ordered by its index set. Suppose $x$ and $y$ in $I - C_{\gamma_1} = G$ are such that $x < y$. If $x$ and $y$ are in some $I(r)$, then since $f$ is by definition increasing on $I(r)$, $f(x) < f(y)$. Suppose now $x$ is in $I(r)$ and $y$ is in $I(l)$. Then $x < y$ implies that $I(r) < I(l)$ and $r < l$. Hence $J(r) < J(l)$. It follows that $f(x) < f(y)$, since $f(x) \in J(r)$ and $f(y) \in J(l)$. We have thus shown that $f$ is increasing on $I - C_{\gamma_1} = G$. Plainly, $g_{1,2}$ maps $G$ bijectively on to $H$.

Next, we show that $g_{1,2}$ *maps* $C_{\gamma_1}$ *into* $C_{\gamma_2}$.

For $x = 0$, $g_{1,2}(0) = 0$.

We now assume $x \neq 0$ and $x \in C_{\gamma_1}$. Recall then that

$$g_{1,2}(x) = \sup\left\{g_{1,2}(y) : y < x \text{ and } y \in G\right\} = \sup\left\{g_{1,2}(y) : y < x \text{ and } y \in [0,1] - C_{\gamma_1}\right\}.$$

Suppose that $g_{1,2}(x) \notin C_{\gamma_2}$. Then for some dyadic rational number $l$, $g_{1,2}(x) \in J(l)$ and since $J(l) = g_{1,2}\left(I(l)\right)$, there exists $x_0$ in $I(l)$ such that $g_{1,2}(x_0) = g_{1,2}(x)$. Then since $I(l)$ is open there exists $y_0$ in $I(l)$ with $y_0 < x_0$ such that $g_{1,2}(y_0) < g_{1,2}(x_0) = g_{1,2}(x)$. By the definition of supremum, there exists $y'$ in $[0,1] - C_{\gamma_1}$ with $y' < x$ and $g_{1,2}(y_0) < g_{1,2}(y') \leq g_{1,2}(x) = g_{1,2}(x_0)$. Since $f$ is increasing on $I - C_{\gamma_1}$, $y_0 < y' < x$. Then since $y_0 \in I(l)$ and so for all $y$ in $I(l)$, $y < x$ for otherwise, if there exists $z$ in $I(l)$ with $z > x$, then $x$ would belong to $(y_0, z) \subseteq I(l) \subseteq [0,1] - C_{\gamma_1}$, contradicting $x \in C_{\gamma_1}$. Now, since $I(l)$ is open, there exists $x'$ in $I(l)$ such that $x' > x_0$. Thus, $g_{1,2}(x') > g_{1,2}(x_0) = g_{1,2}(x)$. Also since $x' < x$, $g_{1,2}(x') \leq \sup\left\{g_{1,2}(y) : y < x \text{ and } y \in [0,1] - C_{\gamma_1}\right\} = g_{1,2}(x)$, contradicting $g_{1,2}(x') > g_{1,2}(x)$. This shows that $g_{1,2}(x) \in C_{\gamma_2}$. Hence, $g_{1,2}$ maps $C_{\gamma_1}$ into $C_{\gamma_2}$.

The function $g_{1,2}$ *is strictly increasing on* [0, 1].

We have already shown that $g_{1,2}$ *is strictly increasing on* $G = I - C_{\gamma_1}$.

Thus, if $x$ and $y$ are in $G$ and $x < y$, then $g_{1,2}(x) < g_{1,2}(y)$.

Suppose now $x \in C_{\gamma_1}$, $y \notin C_{\gamma_1}$ and $x < y$. Then for any

$$z \in G = \bigcup\left\{I(r) : r \text{ a dyadic rational, } r = \sum_{k=1}^{n} \frac{b_k}{2^k}, \text{where } b_k = 0 \text{ or } 1,\ 1 \leq k < n, b_n = 1,\ n \geq 1\right\},$$

$z < x$ implies that $z < y$. Therefore, since $y$ and $z$ are in $G$, $g_{1,2}(z) < g_{1,2}(y)$. Hence,

$g_{1,2}(x) = \sup\{g_{1,2}(z) : z < x \text{ and } z \in G\} \le g_{1,2}(y)$. Since $g_{1,2}(x) \in C_{\gamma_2} = [0,1] - H$, $g_{1,2}(x) \ne g_{1,2}(y)$ and so $g_{1,2}(x) < g_{1,2}(y)$.

Suppose now $x \in C_{\gamma_1}$, $y \notin C_{\gamma_1}$ and $x > y$.

Then $g_{1,2}(y) \le \sup\{g_{1,2}(z) : z < x \text{ and } z \in G\} = g_{1,2}(x)$.

Again, since $g_{1,2}(x) \ne g_{1,2}(y)$ we must have $g_{1,2}(x) > g_{1,2}(y)$.

Suppose $x$ and $y$ are in $C_{\gamma_1}$ and $x < y$. This time we shall use the property of the Cantor set here. Because $C_{\gamma_1}$ is nowhere dense, the intersection $(x, y) \cap \left([0,1] - C_{\gamma_1}\right) \ne \varnothing$. Therefore, there exists $z \in [0,1] - C_{\gamma_1}$ such that $x < z < y$. By what we have just proved $g_{1,2}(x) < g_{1,2}(z) < g_{1,2}(y)$. Therefore, we can conclude that $g_{1,2}(x) < g_{1,2}(y)$. Hence, we have shown that $g_{1,2}$ is strictly increasing on [0, 1].

*The function $g_{1,2}$ is onto and maps $C_{\gamma_1}$ onto $C_{\gamma_2}$.*

Since $g_{1,2}$ maps the complement of $C_{\gamma_1}$ in [0, 1] onto the complement of $C_{\gamma_2}$ in [0, 1], it is sufficient to show that $g_{1,2}$ *maps* $C_{\gamma_1}$ *onto* $C_{\gamma_2}$. By examining the definition of $g_{1,2}$ we can consider a similar function $g_{2,1}$ mapping $C_{\gamma_2}$ into $C_{\gamma_1}$ which is the inverse of $g_{1,2}$. We are going to use this inverse function to construct a pre-image of $y$ in $C_{\gamma_2}$ under $g_{1,2}$. For $y = 0$ in $C_{\gamma_2}$, by definition of $g_{1,2}$, $g_{1,2}(0) = 0$ and 0 is also in $C_{\gamma_1}$. For a fixed $y \ne 0$ in $C_{\gamma_2}$, define the following

$$x = \sup\left\{\left(g_{1,2}\right)^{-1}(z) : z < y \text{ and } z \in H\right\} = \sup\left\{\left(g_{1,2}\right)^{-1}(z) : z < y \text{ and } z \in g_{1,2}(G)\right\}.$$

Note that this is well defined because $H$ is in the image of $g_{1,2}$, the set $\left\{\left(g_{1,2}\right)^{-1}(z) : z < y \text{ and } z \in H\right\}$ is non-empty and bounded above by 1 so that the supremum exists by the completeness property of $\mathbb{R}$.

The same argument for showing that for any $l \ne 0$ in $C_{\gamma_1}$, $g_{1,2}(l)$ is in $C_{\gamma_2}$, applies here to conclude that $x \in C_{\gamma_1}$. Now we claim that $g_{1,2}(x) = y$.

*Note that*

$$\left\{\left(g_{1,2}\right)^{-1}(z) : z < y \text{ and } z \in g_{1,2}(G)\right\} = \left\{x' : g_{1,2}(x') < y \text{ and } g_{1,2}(x') \in g_{1,2}(G)\right\}$$

$$= \{x' : g_{1,2}(x') < y \text{ and } x' \in G\}.$$

Therefore, $x = \sup\{x' : g_{1,2}(x') < y \text{ and } x' \in G\}$.

Next, we claim that for any $z'$ in *G*,

$$z' < x \Leftrightarrow g_{1,2}(z') < y \text{ . ------------------------- (*)}$$

This is deduced as follows.

Let $z'$ in *G*. Then

$z' < x = \sup\{x' : g_{1,2}(x') < y \text{ and } x' \in G\}$

$\Leftrightarrow$ there exists $z_0$ in $\{x' : g_{1,2}(x') < y \text{ and } x' \in G\}$ such that $z' < z_0 \le x$

$\Rightarrow$ there exists $z_0$ in $\{x' : g_{1,2}(x') < y \text{ and } x' \in G\}$ such that $g_{1,2}(z') < g_{1,2}(z_0) \le g_{1,2}(x)$

$\Rightarrow g_{1,2}(z') < y$.

Conversely, if $z'$ is in *G* and $g_{1,2}(z') < y$, then by definition of *x*, $z' \le x$ and so since $z'$ is in *G*, $z' < x$. *This proves our claim.*

Therefore,

$\{g_{1,2}(z') : z' < x \text{ and } z' \in G\} = \{g_{1,2}(z') : g_{1,2}(z') < y \text{ and } z' \in G\}$

$= \{y' : y' < y \text{ and } y' \in g_{1,2}(G) = H\}$

Thus,
$g_{1,2}(x) = \sup\{g_{1,2}(z') : z' < x \text{ and } z' \in G\} = \sup\{y' : y' < y \text{ and } y' \in H\} = y$.

This is seen as follows. Obviously, $g_{1,2}(x) \le y$.

Note that both $g_{1,2}(x)$ and *y* are in $C_{\gamma_2}$. If $g_{1,2}(x) < y$, then since $C_{\gamma_2}$ is nowhere dense, there exists $y_0$ in $I - C_{\gamma_2} = H = g_{1,2}(G)$ such that $g_{1,2}(x) < y_0 < y$. Therefore, there exists $x_0 \in G$ with $g_{1,2}(x_0) = y_0$ and $g_{1,2}(x) < g_{1,2}(x_0) = y_0 < y$. Since $g_{1,2}$ is strictly increasing on *I*, $x < x_0$.

But by (*), $g_{1,2}(x_0) < y$ implies that $x_0 < x$ contradicting $x < x_0$. Therefore, $g_{1,2}(x) = y$. This shows that $g_{1,2}$ maps $C_{\gamma_1}$ onto $C_{\gamma_2}$ and as a consequence, $g_{1,2}$ is onto. Therefore, $g_{1,2}$ is a strictly increasing function mapping *I* onto *I*. By Theorem 3 of *Inverse Function and Continuity*, $g_{1,2}$ is continuous on [0, 1].

**Remark.** The function *g* in Proposition 9, 10 and 13 is just $g_{1,2}$, where $0<\gamma_1<1$ and $\gamma_2=1$.

**Proposition 16.** The function $g_{1,2}$ is differentiable almost everywhere on *I*. $g_{1,2}$ is differentiable on $I-C_{\gamma_1}$ and $\left(g_{1,2}\right)'(x)=\dfrac{\gamma_2}{\gamma_1}$ for *x* in $I-C_{\gamma_1}=G$.

(1) If $\gamma_1=1$ and $0<\gamma_2<1$, then $g_{1,2}$ is not absolutely continuous on *I*.

(2) If $\gamma_2=1$ and $0<\gamma_1\le 1$, then $g_{1,2}$ is absolutely continuous on *I* and $\left(g_{1,2}\right)'(x)=0$ almost everywhere on $C_{\gamma_1}$.

(3) If $0<\gamma_1<1$ and $\gamma_2\ne\gamma_1$ with $0<\gamma_2\le 1$, then $g_{1,2}$ is Lipschitz on *I* and so is absolutely continuous on *I*.

(4) If $0<\gamma_1<1$ and $\gamma_2\ne\gamma_1$ with $0<\gamma_2\le 1$, then $\left(g_{1,2}\right)'(x)=\dfrac{1-\gamma_2}{1-\gamma_1}$ almost everywhere on $C_{\gamma_1}$.

(5) The arc length of the graph of $g_{1,2}$ is $\sqrt{(\gamma_1)^2+(\gamma_2)^2}+\sqrt{(1-\gamma_1)^2+(1-\gamma_2)^2}$ .

**Proof.**

Since $g_{1,2}$ is strictly increasing, it is differentiable almost everywhere on *I*. Since the gradients of the linear parts of $g_{1,2}$ on each of the open intervals in *G* are the same and are equal to $\dfrac{\gamma_2}{\gamma_1}$, $\left(g_{1,2}\right)'(x)=\dfrac{\gamma_2}{\gamma_1}$ for all *x* in $I-C_{\gamma_1}=G$.

**(1)** If $\gamma_1=1$ and $0<\gamma_2<1$, then $m(C_{\gamma_1})=0$ but $m(g_{1,2}(C_{\gamma_1}))=m\left(C_{\gamma_2}\right)=\gamma_2\ne 0$. Therefore, $g_{1,2}$ is not a *N* function and so cannot be absolutely continuous on *I*.

**(2)** If $\delta_2=1$, then $m(g_{1,2}(C_{\gamma_1}))=m(C_{\gamma_2})=m(C_1)=1-1=0$. Moreover, $g_{1,2}$ is differentiable on $I-C_{\gamma_1}$. Therefore, by Theorem 12 part (a) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*", $g_{1,2}$ is absolutely continuous on *I*. By Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, ${g_{1,2}}'(x)=0$ almost everywhere on $C_{\gamma_1}$.

**(3)** For this we are going to use *G*(*n*) to construct a polygonal approximation to $g_{1,2}$ . Define $f_n:I\to\mathbb{R}$, by $f_n(x)=g_{1,2}(x)$ for *x* in *G*(*n*). Now $G(n)=\bigcup\left\{I\left(\dfrac{m}{2^n}\right):1\le m\le 2^n-1\right\}$ and so $f_n(x)=g_{1,2}(x)$ for $x\in I\left(\dfrac{m}{2^n}\right)$, for $1\le m\le 2^n-1$.

Define $f_n(x)$ to be the corresponding end points of $J\left(\frac{m}{2^n}\right)$ for $x$ = end points of $I\left(\frac{m}{2^n}\right)$.

Now $I-G(n)=\left(G(n)\right)^c=\bigcup_{k=1}^{2^n}F(n,k)$ is a disjoint union of closed intervals, each of length $\ell_n=\frac{1}{2^n}\left(1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)\right)$. We define $f_n$ on each $F(n,k)$, $1\le k\le 2^n$ to be given by the linear function or line joining the points on the graph of $g_{1,2}$ given by the image of the end points of $F(n,k)$. Hence, the linear function on each of the closed interval $F(n,k)$ has the same gradient given by

$$\frac{\lambda_n}{\ell_n}=\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)}.$$

More precisely, the polygonal approximation, $f_n(x)$, is given by the points on the graph of $g_{1,2}$, $\left\{\left(a(r),g_{1,2}(a(r))\right),\left(b(r),g_{1,2}(b(r))\right):r\in\left\{\frac{m}{2^n}:1\le m\le 2^n-1\right\}\right\}$, where $G(n)=\bigcup_{k=1}^{n}U(k)=\bigcup\left\{I\left(\frac{m}{2^n}\right):1\le m\le 2^n-1\right\}$ and $I(r)=\left(a(r),b(r)\right)$.

Note that the gradient of the function $f_n$ on the open intervals in $G(n)$ is the same as the gradient of $g_{1,2}$ on the intervals in $G(n)$ and are equal to $\frac{\gamma_2}{\gamma_1}$.

It follows that for any $x\ne y$ in $I$, $\frac{f_n(x)-f_n(y)}{x-y}\le\max\left(\frac{\gamma_2}{\gamma_1},\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)}\right)$.

If $0<\gamma_2\le\gamma_1<1$, then $\frac{\gamma_2}{\gamma_1}\le\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)}$ for all integer $n\ge 1$ so that

$$\frac{f_n(x)-f_n(y)}{x-y}\le\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)},\text{ for all integer } n\ge 1.$$

If $1\ge\gamma_2>\gamma_1>0$, then $\frac{\gamma_2}{\gamma_1}>\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)}$ for all integer $n\ge 1$ so that

$$\frac{f_n(x)-f_n(y)}{x-y}\le\frac{\gamma_2}{\gamma_1},\text{ for all integer } n\ge 1.$$

This means that for any $x$, $y$ in $I$, if $0<\gamma_2\le\gamma_1<1$,

$$\left|f_n(x)-f_n(y)\right|\le\frac{1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\gamma_1\left(1-\left(\frac{2}{3}\right)^n\right)}\left|x-y\right|, \text{ ----------------------------- (10)}$$

and if $1\ge\gamma_2>\gamma_1>0$, then

$$\left|f_n(x)-f_n(y)\right|\le\frac{\gamma_2}{\gamma_1}\left|x-y\right|. \text{ ---------------------------------------- (11)}$$

Plainly, $\left|f_n(x)-f_{n+1}(x)\right|\le\lambda_n=\dfrac{1}{2^n}\left(1-\gamma_2\left(1-\left(\frac{2}{3}\right)^n\right)\right)$ for all $x$ in $I$. This means the sequence of function $\left(f_n(x)\right)$ converges uniformly to a continuous function $f$ on $I$ such that $g_{1,2}(x)=f(x)$ for all $x$ in $G$. It follows that $f=g_{1,2}$ identically on $I$ since $G$ is dense in $I$. Therefore, by taking limit as $n$ tends to infinity in (10) we get, for any $x$, $y$ in $I$,

if $0<\gamma_2\le\gamma_1<1$, then $\left|g_{1,2}(x)-g_{1,2}(y)\right|\le\dfrac{1-\gamma_2}{1-\gamma_1}\left|x-y\right|$, ------------ (12)

and if $1\ge\gamma_2>\gamma_1>0$, then $\left|g_{1,2}(x)-g_{1,2}(y)\right|\le\dfrac{\gamma_2}{\gamma_1}\left|x-y\right|$. -------------- (13)

This means that if $0<\gamma_1<1$ and $\gamma_2\ne\gamma_1$ with $0<\gamma_2\le1$, then $g_{1,2}$ is Lipschitz on $I$ and so $g_{1,2}$ is absolutely continuous on $I$.

**(4)** By Theorem 11 of *Functions of Bounded Variation and Johnson's Indicatrix*, since $g_{1,2}$ is monotone increasing and absolutely continuous,

$$\int_{C_{\gamma_1}}\left(g_{1,2}\right)'(x)dx=m(g_{1,2}(C_{\gamma_1}))=m\left(C_{\gamma_2}\right)=1-\gamma_2=\int_{C_{\gamma_1}}\frac{1-\gamma_2}{1-\gamma_1}dx.$$

Therefore,

$$\int_{C_{\gamma_1}}\left(\left(g_{1,2}\right)'(x)-\frac{1-\gamma_2}{1-\gamma_1}\right)dx=0. \text{ -------------------- (14)}$$

If $0<\gamma_2\le\gamma_1<1$ and if $g_{1,2}$ is differentiable at $x$ in $C_{\gamma_1}$, then it follows by inequality (12) that $\left(g_{1,2}\right)'(x)\le\dfrac{1-\gamma_2}{1-\gamma_1}$. Since $g_{1,2}$ is differentiable almost everywhere and differentiable in $I-C_{\gamma_1}$, $g_{1,2}$ is differentiable almost everywhere in $C_{\gamma_1}$. Suppose $g_{1,2}$ is differentiable in a subset $D$ of $C_{\gamma_1}$ such that the measure $m(C_{\gamma_1}-D)=0$. Then it follows from (14) that

$\int_D \left( \left(g_{1,2}\right)'(x) - \frac{1-\gamma_2}{1-\gamma_1} \right) dx = 0$ and we conclude that $\left(g_{1,2}\right)'(x) = \frac{1-\gamma_2}{1-\gamma_1}$ almost everywhere on *D* and hence on $C_{\gamma_1}$.

If $1 > \gamma_2 > \gamma_1 > 0$, we consider $g_{2,1}$, which is the inverse of $g_{1,2}$. Then by the above proceeding, $g_{2,1}$ is differentiable almost everywhere on $C_{\gamma_2}$ and $\left(g_{2,1}\right)'(x) = \frac{1-\gamma_1}{1-\gamma_2} \neq 0$ almost everywhere on $C_{\gamma_2}$. Suppose $g_{2,1}$ is differentiable on *E* in $C_{\gamma_2}$ and $m(C_{\gamma_2} - E) = 0$. Then $g_{1,2} = \left(g_{2,1}\right)^{-1}$ is differentiable on $g_{2,1}(E) \subseteq C_{\gamma_1}$ *and* $m\left(C_{\gamma_1} - g_{2,1}(E)\right) = mg_{2,1}(C_{\gamma_2} - E) = 0$.
Moreover, for *y* in $g_{2,1}(E) \subseteq C_{\gamma_1}$,

$$g_{1,2}{}'(y) = \left(\left(g_{2,1}\right)^{-1}\right)'(y) = \frac{1}{\left(g_{2,1}\right)'\left(\left(g_{2,1}\right)^{-1}(y)\right)} = \frac{1}{\frac{1-\gamma_1}{1-\gamma_2}} = \frac{1-\gamma_2}{1-\gamma_1}.$$

Hence, $\left(g_{1,2}\right)'(x) = \frac{1-\gamma_2}{1-\gamma_1}$ almost everywhere on $C_{\gamma_1}$.

If $\gamma_2 = 1$ and $1 > \gamma_1 > 0$, then $m\left(g_{1,2}\left(C_{\gamma_1}\right)\right) = m\left(C_{\gamma_2}\right) = m(C_1) = 0$. By Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, $\left(g_{1,2}\right)'(x) = 0$ almost everywhere on $C_{\gamma_1}$. In this case, $g_{1,2}$ is the function *g* in Proposition 10. This completes the proof of this part.

**(5)** The arc length of the graph of the polygonal approximation $f_n$ to $g_{1,2}$ is given by the sum of the lengths of the linear parts over the closure of the open intervals in *U*(*k*) for *k* =1 to *n* plus the total length of the $2^n$ linear segments over the closed intervals in $I - G(n)$ of equal length. The sum of the lengths of the $2^{n-1}$ linear parts of the graph of $f_n$ or $g_{1,2}$ over *U*(*n*) is given by $2^{n-1}\sqrt{\left(\frac{\gamma_1}{3^n}\right)^2 + \left(\frac{\gamma_2}{3^n}\right)^2} = \frac{1}{2}\left(\frac{2}{3}\right)^n \sqrt{\left(\gamma_1\right)^2 + \left(\gamma_2\right)^2}$. The total length of the $2^n$ linear segments over the closed intervals in $I - G(n)$ is given by

$$2^n\sqrt{\left(\ell_n\right)^2 + \left(\lambda_n\right)^2} = 2^n\sqrt{\left(\frac{1}{2^n}\left(1-\gamma_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)\right)^2 + \left(\frac{1}{2^n}\left(1-\gamma_2\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)\right)^2}$$

$$= \sqrt{\left(1-\gamma_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2 + \left(1-\gamma_2\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2}.$$

Therefore, the arc length of the graph of the polygonal approximation $f_n$ to $g_{1,2}$ is given by

$$\frac{1}{2}\sqrt{(\gamma_1)^2+(\gamma_2)^2}\sum_{k=1}^{n}\left(\frac{2}{3}\right)^k+\sqrt{\left(1-\gamma_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2+\left(1-\gamma_2\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2}$$

$$=\sqrt{(\gamma_1)^2+(\gamma_2)^2}\left(1-\left(\frac{2}{3}\right)^n\right)+\sqrt{\left(1-\gamma_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2+\left(1-\gamma_2\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2}\ .$$

Letting *n* tend to infinity, we obtain the arc length of the graph of $g_{1,2}$ as

$$\sqrt{(\gamma_1)^2+(\gamma_2)^2}+\sqrt{(1-\gamma_1)^2+(1-\gamma_2)^2}\ .$$

We add that for the case when $g_{1,2}$ is absolutely continuous, i.e., when $0<\gamma_1<1$ and $\gamma_2\neq\gamma_1$ with $0<\gamma_2\leq 1$, we may use the arc length formula as usual to obtain the arc length as follows.

$$\int_I\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx=\int_{I-C_{\gamma_1}}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx+\int_{C_{\gamma_1}}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx$$

$$=\int_{I-C_{\gamma_1}}\sqrt{1+\left(\frac{\gamma_2}{\gamma_1}\right)^2}\,dx+\int_{C_{\gamma_1}}\sqrt{1+\left(\frac{1-\gamma_2}{1-\gamma_1}\right)^2}\,dx,$$

since $\left(g_{1,2}\right)'(x)=\dfrac{\gamma_2}{\gamma_1}$ on $G$

and $\left(g_{1,2}\right)'(x)=\dfrac{1-\gamma_2}{1-\gamma_1}$ almost everywhere on $C_\gamma$,

$$=\frac{\sqrt{(\gamma_1)^2+(\gamma_2)^2}}{\gamma_1}m(G)+\frac{\sqrt{(1-\gamma_1)^2+(1-\gamma_2)^2}}{1-\gamma_1}m(C_{\gamma_1})$$

$$=\sqrt{(\gamma_1)^2+(\gamma_2)^2}+\sqrt{(1-\gamma_1)^2+(1-\gamma_2)^2}\ .$$

**Remark**. We may actually use the formula given by Theorem 9 in *Arc Length, Functions of Bounded Variation and Total Variation*, to determine the arc length of $g_{1,2}$.

We state the result as follows:

**Proposition 17.** Suppose $f:[a,b]\to\mathbb{R}$ is an increasing and continuous function. The arc length of the graph of *f* is given by

$$\int_{[a,b]}\sqrt{1+\left(f'(x)\right)^2}\,dx+T_h[a,b],$$

where $h(x)=f(x)-\int_a^x f'(x)dx$ is an increasing continuous function and is the singular part of *f* in the Lebesgue decomposition of *f* and $T_h[a,b]$ is the total variation of *h* on [*a*, *b*]. Moreover, $T_h[a,b]=h(b)-h(a)$.

**Proof.** This is a specialization of Theorem 9 in *Arc Length, Functions of Bounded Variation and Total Variation* to a continuous increasing function. By Theorem 15 of *Arc Length, Functions of Bounded Variation and Total Variation*, $h(x)=f(x)-\int_a^x f'(x)dx$ is a continuous increasing function and so its total variation $T_h[a,b]$ is equal to $h(b)-h(a)$.

**Remark.** If $\gamma_1=1$ , then $m\left(C_{\gamma_1}\right)=0$ and $g_{1,2}:[0,1]\to[0,1]$ is not absolutely continuous.

Since it is continuous and increasing on [0, 1], the arc length of its graph is given by

$$\int_{[0,1]}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx+h(1),$$

where $h(x)=g_{1,2}(x)-\int_0^x\left(g_{1,2}\right)'(x)dx$.

Now,

$$\int_{[0,1]}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx=\int_{[0,1]-C_1}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx+\int_{C_1}\sqrt{1+\left(\left(g_{1,2}\right)'(x)\right)^2}\,dx$$

$$=\int_{[0,1]-C_1}\sqrt{1+\left(\frac{\gamma_2}{1}\right)^2}\,dx+0=\sqrt{1+\left(\gamma_2\right)^2}\,m\left([0,1]-C_1\right)=\sqrt{1+\left(\gamma_2\right)^2},$$

and

$$h(1)=g_{1,2}(1)-\int_0^1\left(g_{1,2}\right)'(x)dx=1-\int_{I-C_1}\left(g_{1,2}\right)'(x)dx-\int_{C_1}\left(g_{1,2}\right)'(x)dx$$

$$=1-\int_{I-C_1}\gamma_2dx-0=1-\gamma_2m(I-C_1)=1-\gamma_2.$$

Hence, the arc length of the graph of $g_{1,2}$ is $1-\gamma_2+\sqrt{1+\left(\gamma_2\right)^2}$.

**Proposition 18.** Let the function, $g_{1,2}:I\to I$ , be defined as above mapping Cantor set $C_{\gamma_1}$ linearly and bijectively onto $C_{\gamma_2}$, where $0<\gamma_1,\gamma_2\le 1$ and $\gamma_1\ne\gamma_2$. Then $\int_{I-C_{\gamma_1}}g_{1,2}(x)dx=\frac{\gamma_1}{2}$ and $\int_{C_{\gamma_1}}g_{1,2}(x)dx=\frac{1-\gamma_1}{2}$.

**Proof.**

Let $G=[0,1]-C_{\gamma_1}=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}}I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $I(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\gamma_1}{3^n}$, to be deleted in stage $n$ in the construction of $C_{\gamma_1}$. Let $H=[0,1]-C_{\gamma_2}=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}}J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $J(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\gamma_2}{3^n}$, to be deleted in stage $n$ in the construction of $C_{\gamma_2}$. Let $G(n)==\bigcup_{k=1}^{n}U(k)$ and $H(n)=\bigcup_{k=1}^{n}V(k)$. Then $G=\bigcup_{k=1}^{\infty}G(k)$ and $H=\bigcup_{k=1}^{\infty}H(k)$. $I-G(n)=F(n,1)\cup F(n,2)\cup\cdots\cup F(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\ell_n=\frac{1}{2^n}\left(1-\gamma_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)$ and $I-H(n)=K(n,1)\cup K(n,2)\cup\cdots\cup K(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\lambda_n=\frac{1}{2^n}\left(1-\gamma_2\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)$.

Now we examine the indexing of $U(n)$ and $V(n)$. We note that

$$\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}=\left\{\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}:\varepsilon_j=0\text{ or }1\right\}.$$

If we let $J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)=J\left(\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}\right)=(c(r),d(r))$, where $r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}$, then

$$c(r)=\sum_{j=1}^{n-1}\left(\left(\lambda_j+\frac{\gamma_2}{3^j}\right)\varepsilon_j+\lambda_n\right).$$

Therefore,

$$\int_{U(n)}g_{1,2}(x)dx=\frac{\gamma_1}{3^n}\times\sum_{r\in\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}}c(r)+2^{n-1}\left(\frac{1}{2}\times\frac{\gamma_1}{3^n}\times\frac{\gamma_2}{3^n}\right)$$

$$=\frac{\gamma_1}{3^n}\times\sum_{r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}}\left(\sum_{j=1}^{n-1}\left(\left(\lambda_j+\frac{\gamma_2}{3^j}\right)\varepsilon_j+\lambda_n\right)\right)+2^{n-2}\frac{\gamma_1}{3^n}\times\frac{\gamma_2}{3^n}$$

$$=\frac{\gamma_1}{3^n}\times\left(2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\gamma_2}{3^j}+2^{n-1}\lambda_n\right)+2^{n-2}\frac{\gamma_1}{3^n}\times\frac{\gamma_2}{3^n}. \quad \text{------------------------- (*2)}$$

But $\frac{\gamma_2}{3^j} = \lambda_{j-1} - 2\lambda_j$ for $1 \le j \le n$ . For $j = 1$, $\lambda_{j-1} = \lambda_0$ is set to be 1.

Therefore,

$$
\begin{aligned}
&2^{n-2}\sum_{j=1}^{n-1}\lambda_j + 2^{n-2}\sum_{j=1}^{n-1}\frac{\gamma_2}{3^j} + 2^{n-1}\lambda_n + 2^{n-2}\cdot\frac{\gamma_2}{3^n} \\
&= 2^{n-2}\sum_{j=1}^{n-1}\lambda_j + 2^{n-2}\sum_{j=1}^{n-1}\left(\lambda_{j-1} - 2\lambda_j\right) + 2^{n-1}\lambda_n + 2^{n-2}\cdot\left(\lambda_{n-1} - 2\lambda_n\right) \\
&= 2^{n-2}\sum_{j=1}^{n-1}\lambda_j - 2^{n-1}\sum_{j=1}^{n-1}\lambda_j + 2^{n-2}\lambda_0 + 2^{n-2}\sum_{j=1}^{n-2}\lambda_j + 2^{n-2}\lambda_{n-1} = 2^{n-2}\lambda_0 = 2^{n-2}.
\end{aligned}
$$

It follows then from (*2) that

$$\int_{U(n)} g_{1,2}(x)dx = \frac{\gamma_1}{3^n}\times 2^{n-2} = \frac{\gamma_1}{2}\frac{2^{n-1}}{3^n}.$$

Hence,

$$
\begin{aligned}
\int_G g_{1,2}(x)dx = \int_{I-C_{\gamma_1}} g_{1,2}(x)dx &= \sum_{n=1}^{\infty}\int_{U(n)} g_{1,2}(x)dx \\
&= \frac{\gamma_1}{2}\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n} = \frac{\gamma_1}{2}.
\end{aligned}
$$

Therefore, as $\int_I g_{1,2}(x)dx = \frac{1}{2}$, $\int_{C_{\gamma_1}} g_{1,2}(x)dx = \frac{1}{2} - \int_{I-C_{\gamma_1}} g_{1,2}(x)dx = \frac{1-\gamma_1}{2}$.

**Remark.**

Proposition 14 is a special case of Proposition 17.

**1.** The function, $g_{1,2}$, defined above satisfies $g_{1,2}(x) + g_{1,2}(1-x) = 1$ for all $x$ in $I$. This has exactly the same proof as shown above for $g$ in Proposition 13. Consequently, $\int_I g_{1,2}(x)dx = \frac{1}{2}$.

**2**. When $\gamma_1 = 1$, $\int_{C_{\gamma_1}} g_{1,2}(x)dx = 0$ trivially as $m(C_{\gamma_1}) = m(C_1) = 0$. When $\gamma_2 = 1$, $\int_{C_{\gamma_1}} g_{1,2}(x)dx = \frac{1-\gamma_1}{2}$ follows from Proposition 14.

**3.** When $0 < \gamma_1, \gamma_2 < 1$ and $\gamma_1 \ne \gamma_2$, we may deduce that $\int_{I-C_{\gamma_1}} g_{1,2}(x)dx = \frac{\gamma_1}{2}$ as follows, Factorize $g_{1,2}$ as $h \circ g$ , where $g$ is the Cantor Lebesgue type function defined before, mapping $C_{\gamma_1}$ on to $C_1$ and $h$ is the Cantor Lebesgue type function, mapping $C_1$ on to $C_{\gamma_2}$.

By Proposition 10, $g$ is absolutely continuous. Therefore, by Theorem 8 of *Change of Variables Theorems*,

$$\frac{1}{2} = \int_I h(x)dx = \int_I h \circ g(x) g'(x)dx = \int_I g_{1,2}(x) g'(x)dx$$
$$= \int_{C_{\gamma_1}} g_{1,2}(x) g'(x)dx + \int_{I-C_{\gamma_1}} g_{1,2}(x) g'(x)dx$$
$$= 0 + \int_{I-C_{\gamma_1}} g_{1,2}(x) \cdot \frac{1}{\gamma_1} dx = \frac{1}{\gamma_1} \int_{I-C_{\gamma_1}} g_{1,2}(x)dx ,$$

since $g'(x) = 0$ almost everywhere on $C_{\gamma_1}$ and $g'(x) = \frac{1}{\gamma_1}$ on $I - C_{\gamma_1}$.

Therefore, $\frac{1}{\gamma_1} \int_{I-C_{\gamma_1}} g_{1,2}(x)dx = \frac{1}{2}$ and so $\int_{I-C_{\gamma_1}} g_{1,2}(x)dx = \frac{\gamma_1}{2}$ and $\int_{C_{\gamma_1}} g_{1,2}(x)dx = \frac{1}{2} - \frac{\gamma_1}{2} = \frac{1-\gamma_1}{2}$.

**4.** Note that $g_{1,2} = h \circ g$ as in part 3 above is absolutely continuous even though $h$ is not absolutely continuous.

**Two Families of Cantor Sets**

For $0 < \delta \leq 1$, let $D_\delta$ be the cantor sets constructed in a similar fashion as $C_\delta$ and described in my article, *The Construction of cantor Sets*, as shown below.

In the first stage, we delete the middle open interval with length $\frac{\delta}{2}$ from [0, 1]. Following the construction for $C_\delta$, we denote this open interval by $I(1,1)$. Then the complement of this middle interval is 2 disjoint closed interval each of length $\frac{1}{2}\left(1 - \frac{\delta}{2}\right)$. We denote the open deleted interval by $I(1,1) = (a(1,1), b(1,1))$. As for the case of $C_\delta$, we denote the closed interval in the complement by $J(1,1)$ and $J(1,2)$, where the closed interval $J(1,2)$ is to the right $J(1,1)$ , meaning it is ordered in such a way that every point of $J(1,2)$ is bigger than any point in $J(1,1)$.

Then at the second stage we delete the middle open interval of length $\frac{\delta}{2^3}$ from each of the 2 remaining closed intervals. Thus, there are 2 open intervals to be deleted and they are $I(2,1) = (a(2,1), b(2,1))$ and $I(2,2) = (a(2,2), b(2,2))$. As for the case of $C_\delta$, these two open intervals are ordered by the second index. Hence, we are left with 4 = $2^2$ remaining closed intervals, $J(2,1)$, $J(2,2)$, $J(2,3)$ and $J(2,2^2)$ each of length $\frac{1}{2^2}\left(1-\delta\left(1-\left(\tfrac{1}{2}\right)^2\right)\right)$. Let $U(1)=I(1,1)$, $U(2)=I(2,1)\cup I(2,2)$, $G(1)=U(1)$, $G(2)=U(1)\cup U(2)$. Then $I-G(1)=J(1,1)\cup J(1,2)$, $I-G(2)=J(2,1)\cup J(2,2)\cup J(2,3)\cup J(2,2^2)$.

At stage *n* delete the middle open interval of length $\frac{\delta}{2^{2n-1}}$ from each of the $2^{n-1}$ remaining closed intervals, $J(n-1,1),J(n-1,2),\cdots,J(n-1,2^{n-1})$, each of length $\frac{1}{2^{n-1}}\left(1-\delta\left(1-\left(\tfrac{1}{2}\right)^{n-1}\right)\right)$. Denote these open intervals by $I(n,1),I(n,2),\cdots,I(n,2^{n-1})$. Then this resulted in the remaining $2^n$ closed intervals, $J(n,1),J(n,2),\cdots,J(n,2^n)$, each of length $\ell_n=\frac{1}{2^n}\left(1-\delta\left(1-\left(\tfrac{1}{2}\right)^n\right)\right)$. Let $U(n)=\bigcup_{k=1}^{2^{n-1}}I(n,k)$ and $G(n)=\bigcup_{k=1}^{n}U(k)=\bigcup_{k=1}^{n}\bigcup_{j=1}^{2^{k-1}}I(k,j)$.

Observe that $I-G(n)=J(n,1)\cup J(n,2)\cup\cdots\cup J(n,2^n)$.

Note that *G*(*n*) consists of $2^n-1$ disjoint open intervals. The total length of the intervals in *G*(*n*) is given by

$$\frac{\delta}{2}+2\frac{\delta}{2^3}+2^2\frac{\delta}{2^5}+\cdots+2^{n-1}\frac{\delta}{2^{2n-1}}=\frac{\delta}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2\cdots+\left(\frac{1}{2}\right)^{n-1}\right)$$
$$=\delta\left(1-\frac{1}{2^n}\right).$$

Note that $G(n)\subseteq G(n+1)$ and $G(n+1)=G(n)\cup U(n+1)$.

Let $G=\bigcup_{k=1}^{\infty}G(k)=\bigcup_{k=1}^{\infty}U(k)$. Thus, the measure of *G* is the total length of all the *U*(*k*), that is, $m(G)=\lim_{n\to\infty}\delta\left(1-\frac{1}{2^n}\right)=\delta$.

The Cantor set $D_\delta$ is defined by $D_\delta=I-G$. Hence the measure of $D_\delta$ is given by $m(I)-m(G)=1-\delta\geq 0$.

We re-index $I(n,k_n)$, $1\leq k_n\leq 2^{n-1}$, $1\leq n<\infty$ as for the deleted open intervals for $C_\delta$.

We define the Cantor Lebesgue function $g_{D_\delta}$ associated with $D_\delta$ in exactly the same manner as for $C_\delta$, For $x\in G$, $x\in I(r)$, $g_{D_\delta}(x)=r$.

All the properties that we have proved for $C_\delta$ apply to $D_\delta$ and its associated Cantor Lebesgue function, $g_{D_\delta}$.

In summary, we have the following theorems.

**Theorem 19.** The Cantor set $D_\delta$ ($0 < \delta \le 1$) is

(1) compact,

(2) nowhere dense, i.e., it contains no open intervals,

(3) its own boundary points,

(4) perfect, i.e., it is its own set of accumulation points,

(5) totally disconnected and

(6) between any two points in $D_\delta$, there is an open interval not contained in $D_\delta$ .

**Theorem 20.** The Cantor Lebesgue function, $g_{D_\delta}$, associated with the Cantor set $D_\delta$ ($0 < \delta \le 1$) is increasing and continuous and maps $D_\delta$ onto $I = [0,1]$.

(1) If $0 < \delta < 1$, then the associated Cantor Lebesgue function $g_{D_\delta}$ is Lipschitz with constant $\frac{1}{1-\delta}$ and so is absolutely continuous on [0, 1]; if $\delta = 1$, then $g_{D_\delta}$ is singular and not absolutely continuous.

(2) $g_{D_\delta}$ satisfies the relation $g_{D_\delta}(x) + g_{D_\delta}(1-x) = 1$, for all $x$ in [0,1].

(3) The arc length of the graph of the Cantor Lebesgue function, $g_{D_\delta} : [0,1] \to [0,1]$, for $D_\delta$, is $\delta + \sqrt{1+(1-\delta)^2}$ .

(4) $g_{D_\delta}$ is differentiable almost everywhere on [0,1]; $\left(g_{D_\delta}\right)'(x) = 0$ for $x \in [0,1] - D_\delta$; if $0 < \delta < 1$, $\left(g_{D_\delta}\right)'(x) = \frac{1}{1-\delta}$ almost everywhere on $D_\delta$.

(5) $\int_0^1 g_{D_\delta}(x)dx = \frac{1}{2}$, $\int_{I-D_\delta} g_{D_\delta}(x)dx = \frac{\delta}{2}$ and $\int_{D_\delta} g_{D_\delta}(x)dx = \frac{1-\delta}{2}$.

The proof for Theorem 19 and 20 is exactly the same for the Cantor set $C_\gamma$ and its associated Cantor Lebesgue function. The proof for part (2) of Theorem 20 is similar to that of Proposition 13.

Let $g_{D_{\delta_1},D_{\delta_2}}$ be the canonical Cantor like function defined similarly as $g_{1,2}$ mapping the cantor set $D_{\delta_1}$ onto $D_{\delta_2}$.

We have analogous results for $g_{D_{\delta_1},D_{\delta_2}}$ as for $g_{1,2}$.

**Theorem 21.** For $0<\delta_1,\delta_2\leq 1$ and $\delta_1\neq\delta_2$, the associated Cantor like function $g_{D_{\delta_1},D_{\delta_2}}:I\to I$ is strictly increasing and continuous and maps the Cantor set $D_{\delta_1}$ bijectively onto $D_{\delta_2}$. $g_{D_{\delta_1},D_{\delta_2}}$ is differentiable almost everywhere on *I*.

(1) $g_{D_{\delta_1},D_{\delta_2}}{}'(x)=\dfrac{\delta_2}{\delta_1}$ for *x* in $I-D_{\delta_1}$.

(2) If $\delta_1=1$ and $0<\delta_2<1$, then $g_{D_{\delta_1},D_{\delta_2}}$ is not absolutely continuous on *I*.

(3) If $\delta_2=1$ and $0<\delta_1\leq 1$, then $g_{D_{\delta_1},D_{\delta_2}}$ is absolutely continuous on *I* and $g_{D_{\delta_1},D_{\delta_2}}{}'(x)=0$ almost everywhere on $D_{\delta_1}$.

(4) If $0<\delta_1<1$ and $\delta_2\neq\delta_1$ with $0<\delta_2\leq 1$, then $g_{D_{\delta_1},D_{\delta_2}}$ is Lipschitz on *I* and so is absolutely, continuous on *I and* $g_{D_{\delta_1},D_{\delta_2}}{}'(x)=\dfrac{1-\delta_2}{1-\delta_1}$ almost everywhere on $D_{\delta_1}$.

(5) The arc length of the graph of $g_{D_{\delta_1},D_{\delta_2}}$ is $\sqrt{(\delta_1)^2+(\delta_2)^2}+\sqrt{(1-\delta_1)^2+(1-\delta_2)^2}$ .

(6) $\int_{I-D_{\delta_1}} g_{D_{\delta_1},D_{\delta_2}}(x)dx=\dfrac{\delta_1}{2}$ and $\int_{D_{\delta_1}} g_{D_{\delta_1},D_{\delta_2}}(x)dx=\dfrac{1-\delta_1}{2}$.

(7) $g_{D_{\delta_1},D_{\delta_2}}(x)+g_{D_{\delta_1},D_{\delta_2}}(1-x)=1$ for all *x* in *I*.

**Proof.** We omit the proof as it is exactly similar to the case for $g_{1,2}$ .

Now we consider the case of map from one cantor set of a family to another cantor set in another family.

Let $g_{C_{\delta_1},D_{\delta_2}}$ be the canonical Cantor like function defined similarly as $g_{1,2}$ mapping the cantor set $C_{\delta_1}$ onto the Cantor set $D_{\delta_2}$.

**Proposition 22.** The function, $g_{C_{\delta_1},D_{\delta_2}}:I\to I$ , as defined above is strictly increasing and continuous and maps the Cantor set $C_{\delta_1}$ bijectively onto $D_{\delta_2}$, where $0<\delta_1,\delta_2\leq 1$. The function $g_{C_{\delta_1},D_{\delta_2}}$ satisfies $g_{C_{\delta_1},D_{\delta_2}}(x)+g_{C_{\delta_1},D_{\delta_2}}(1-x)=1$ for all *x* in *I*.

**Proof.** The proof is exactly the same as for Proposition 15.

Note that $g_{D_{\delta_2},C_{\delta_1}}$ is the inverse to $g_{C_{\delta_1},D_{\delta_2}}:I\to I$.

**Theorem 23.** $g_{C_{\delta_1},D_{\delta_2}} : I \to I$ is differentiable almost everywhere on *I*. $g_{C_{\delta_1},D_{\delta_2}}$ is differentiable on $[0,1]-C_{\delta_1}$. For *x* in $I - C_{\delta_1}$, if $x \in I(r)$ and

$r = \sum_{k=1}^{n} \frac{b_k}{2^k}$, where $b_k = 0$ or 1, $1 \le k < n, b_n = 1, n \ge 1$, then $g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \frac{3^n}{2^{2n-1}} \frac{\delta_2}{\delta_1} = 2\left(\frac{3}{4}\right)^n \frac{\delta_2}{\delta_1}$.

(1) If $\delta_1 = 1$ and $0 < \delta_2 < 1$, then $g_{C_{\delta_1},D_{\delta_2}}$ is not absolutely continuous on *I.*

(2) If $\delta_2 = 1$ and $0 < \delta_1 \le 1$ then $g_{C_{\delta_1},D_{\delta_2}}$ is absolutely continuous on [0, 1] and $g_{C_{\delta_1},D_{\delta_2}}{}'(x) = 0$ almost everywhere on $C_{\delta_1}$.

(3) If $0 < \delta_1 < 1$ and $0 < \delta_2 \le 1$, then $g_{C_{\delta_1},D_{\delta_2}}$ is Lipschitz on *I* and so is absolutely continuous on *I* and $g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \frac{1-\delta_2}{1-\delta_1}$ almost everywhere on $C_{\delta_1}$.

(4) The arc length of the graph of $g_{C_{\delta_1},D_{\delta_2}}$ is

$$\sqrt{(1-\delta_1)^2 + (1-\delta_2)^2} + \frac{1}{2}\sum_{k=1}^{\infty} \left(\tfrac{2}{3}\right)^k \sqrt{(\delta_1)^2 + 4(\delta_2)^2 \left(\tfrac{3}{4}\right)^{2k}} \ .$$

(5) $\int_{I-C_{\delta_1}} g_{C_{\delta_1},D_{\delta_2}}(x)dx = \frac{\delta_1}{2}$ and $\int_{C_{\delta_1}} g_{C_{\delta_1},D_{\delta_2}}(x)dx = \frac{1-\delta_1}{2}$.

**Proof.** Since $g_{C_{\delta_1},D_{\delta_2}}$ is strictly increasing, it is differentiable almost everywhere on [0,1].

Let $G = I - C_{\delta_1}$

$$= \bigcup \left\{ I(r) : r \text{ a dyadic ratonal, } r = \sum_{k=1}^{n} \frac{b_k}{2^k}, \text{where } b_k = 0 \text{ or } 1,\ 1 \le k < n, b_n = 1,\ n \ge 1 \right\},$$

where $I(r)$ are the open intervals in *G* to be deleted to construct $C_{\delta_1}$.

Let $H = I - D_{\delta_2}$

$$= \bigcup \left\{ J(r) : r \text{ a dyadic rational, } r = \sum_{k=1}^{n} \frac{b_k}{2^k}, \text{where } b_k = 0 \text{ or } 1,\ 1 \le k < n, b_n = 1,\ n \ge 1 \right\},$$

where $J(r)$ are the open intervals in *H* to be deleted to construct $D_{\delta_2}$.

Note that if $r = \sum_{k=1}^{n} \frac{b_k}{2^k}$, where $b_k = 0$ or 1, $1 \le k < n, b_n = 1, n \ge 1$, then the length of *I*(*r*) is $\frac{\delta_1}{3^n}$ and that of *J*(*r*) is $\frac{\delta_2}{2^{2n-1}}$.

Plainly, if $x \in I(r)$ and $r = \sum_{k=1}^{n} \frac{b_k}{2^k}$, where $b_k = 0$ or 1, $1 \le k < n, b_n = 1, n \ge 1$, then

$$g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \frac{\text{length of } J(r)}{\text{length of } I(r)} = \frac{\frac{\delta_2}{2^{2n-1}}}{\frac{\delta_1}{3^n}} = \frac{3^n}{2^{2n-1}}\frac{\delta_2}{\delta_1} = 2\left(\frac{3}{4}\right)^n \frac{\delta_2}{\delta_1}.$$

(1) If $\delta_1 = 1$ and $0 < \delta_2 < 1$, then $m(g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1})) = m(g_{C_{\delta_1},D_{\delta_2}}(C_1)) = m(D_{\delta_2}) = 1 - \delta_2 > 0$. As $m(C_1) = 0$, $g_{C_{\delta_1},D_{\delta_2}}$ cannot be a *N* function. Consequently, $g_{C_{\delta_1},D_{\delta_2}}$ is not absolutely continuous on *I*.

(2) If $\delta_2 = 1$, then $m(g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1})) = m(D_{\delta_2}) = m(D_1) = 1 - 1 = 0$. Moreover, $g_{C_{\delta_1},D_{\delta_2}}$ is differentiable on $I - C_{\delta_1}$ and $g_{C_{\delta_1},D_{\delta_2}}{}'(x) \le 2\frac{\delta_2}{\delta_1} = \frac{2}{\delta_1}$ for all *x* in $I - C_{\delta_1}$. Therefore, by Theorem 12 part (a) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $g_{C_{\delta_1},D_{\delta_2}}$ is absolutely continuous on *I*. By Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, $g_{C_{\delta_1},D_{\delta_2}}{}'(x) = 0$ almost everywhere on $C_{\delta_1}$.

(3) Suppose $0 < \delta_1 < 1$ and $0 < \delta_2 \le 1$. Let $g_n$ be the polygonal approximation of $g_{C_{\delta_1},D_{\delta_2}}$ determined by the points on the graph given by the end points of the open intervals in $G(n) = \bigcup_{j=1}^{n} U(j) = \bigcup_{k=1}^{2^n-1} I\left(\frac{k}{2^n}\right)$. As the gradient of the graph of $g_n$ over each open interval in *U*(*j*) is $2\left(\frac{3}{4}\right)^j \frac{\delta_2}{\delta_1}$ and over each closed interval in $I - G(n)$ is $\dfrac{1-\delta_2\left(1-\frac{1}{2^n}\right)}{1-\delta_1\left(1-\left(\frac{2}{3}\right)^n\right)}$, for $x \ne y$,

$$0 \le \frac{g_n(x) - g_n(y)}{x-y}$$

$$\le \max\left\{2\left(\frac{3}{4}\right)\frac{\delta_2}{\delta_1}, 2\left(\frac{3}{4}\right)^2\frac{\delta_2}{\delta_1}, \cdots, 2\left(\frac{3}{4}\right)^n\frac{\delta_2}{\delta_1}, \frac{1-\delta_2\left(1-\frac{1}{2^n}\right)}{1-\delta_1\left(1-\left(\frac{2}{3}\right)^n\right)}\right\} = \max\left\{\frac{3}{2}\frac{\delta_2}{\delta_1}, \frac{1-\delta_2\left(1-\frac{1}{2^n}\right)}{1-\delta_1\left(1-\left(\frac{2}{3}\right)^n\right)}\right\}.$$

Therefore, taking limit as *n* tends to infinity we have, for $x \ne y$,

$$0 \le \frac{g_{C_{\delta_1},D_{\delta_2}}(x) - g_{C_{\delta_1},D_{\delta_2}}(y)}{x-y} \le \max\left\{\frac{3}{2}\frac{\delta_2}{\delta_1}, \frac{1-\delta_2}{1-\delta_1}\right\}. \text{ ------------------- (15)}$$

It follows that $\left|g_{C_{\delta_1},D_{\delta_2}}(x) - g_{C_{\delta_1},D_{\delta_2}}(y)\right| \le \max\left\{\frac{3}{2}\frac{\delta_2}{\delta_1}, \frac{1-\delta_2}{1-\delta_1}\right\}|x-y|$ for all *x*, *y* in *I*. Hence, $g_{C_{\delta_1},D_{\delta_2}}$ is Lipschitz and so is absolutely continuous on *I*.

The case when $\delta_2 = 1$ has already been dealt with in part (2).

Now we assume $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$. Note that $\frac{3}{2}\frac{\delta_2}{\delta_1} \le \frac{1-\delta_2}{1-\delta_1} \Leftrightarrow \delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$.

Thus, if $\delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$, it follows from (15) that for $x \ne y$,

$$\frac{g_{C_{\delta_1},D_{\delta_2}}(x) - g_{C_{\delta_1},D_{\delta_2}}(y)}{x-y} \le \frac{1-\delta_2}{1-\delta_1}.$$

Therefore, if $\delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$, $\left|g_{C_{\delta_1},D_{\delta_2}}(x) - g_{C_{\delta_1},D_{\delta_2}}(y)\right| \le \frac{1-\delta_2}{1-\delta_1}|x-y|$ for any *x* and *y* in *I*.

Hence, if $\delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$, $g_{C_{\delta_1},D_{\delta_2}}$ is Lipschitz with constant $\frac{1-\delta_2}{1-\delta_1}$.

Therefore, if $\delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$, since $g_{C_{\delta_1},D_{\delta_2}}$ is strictly increasing and absolutely continuous, by Theorem 11, *Functions of Bounded Variation and Johnson's Indicatrix*,

$$\int_{C_{\delta_1}} g_{C_{\delta_1},D_{\delta_2}}{}'(x)dx = m\left(g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1})\right) = m\left(D_{\delta_2}\right) = 1-\delta_2 = \int_{C_{\delta_1}} \frac{1-\delta_2}{1-\delta_1}dx.$$

It follows that if $\delta_1 + 2\frac{\delta_1}{\delta_2} \ge 3$, $\int_{C_{\delta_1}}\left(\frac{1-\delta_2}{1-\delta_1} - g_{C_{\delta_1},D_{\delta_2}}{}'(x)\right)dx = 0$. As

$\frac{g_{C_{\delta_1},D_{\delta_2}}(x) - g_{C_{\delta_1},D_{\delta_2}}(y)}{x-y} \le \frac{1-\delta_2}{1-\delta_1}$, for *x* in $C_{\delta_1}$ and $g_{C_{\delta_1},D_{\delta_2}}$ is differentiable at *x*, then

$g_{C_{\delta_1},D_{\delta_2}}{}'(x) \le \frac{1-\delta_2}{1-\delta_1}$. Since $g_{C_{\delta_1},D_{\delta_2}}$ is differentiable almost everywhere on [0,1],

$g_{C_{\delta_1},D_{\delta_2}}{}'(x) \le \frac{1-\delta_2}{1-\delta_1}$ almost everywhere on $C_{\delta_1}$. Therefore, $g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \frac{1-\delta_2}{1-\delta_1}$ almost everywhere on $C_{\delta_1}$.

If $\delta_1 + 2\frac{\delta_1}{\delta_2} < 3$, we shall factor $g_{C_{\delta_1},D_{\delta_2}}$ as $g_{C_{\delta_3},D_{\delta_2}} \circ g_{C_{\delta_1},C_{\delta_3}}$, where $\delta_3 + 2\frac{\delta_3}{\delta_2} \ge 3$.

As the function $h(x) = x + 2\frac{x}{\delta_2}$ is continuous on [0, 1], *h*(0)= 0 and $h(1) = 1 + \frac{2}{\delta_2} > 3$, if $\delta_1 + 2\frac{\delta_1}{\delta_2} < 3$, by the Intermediate Value Theorem, there exists, $\delta_1 < \delta_3 < 1$ such that $h(\delta_3) = \delta_3 + 2\frac{\delta_3}{\delta_2} = 3$. Choose such a $\delta_3$ for the factorization. Then $g_{C_{\delta_3},D_{\delta_2}}$ is absolutely continuous on *I* and

$$g_{C_{\delta_3},D_{\delta_2}}{}'(x) = \frac{1-\delta_2}{1-\delta_3} \text{ almost everywhere on } C_{\delta_3}.$$

As $0 < \delta_1 < \delta_3 < 1$, by Proposition 16, $g_{C_{\delta_1},C_{\delta_3}}$ is Lipschitz and

$$g_{C_{\delta_1},C_{\delta_3}}{}'(x) = \frac{1-\delta_3}{1-\delta_1} \text{ almost everywhere on } C_{\delta_1}.$$

Since $g_{C_{\delta_1},C_{\delta_3}}$, $g_{C_{\delta_3},D_{\delta_2}}$ and $g_{C_{\delta_3},D_{\delta_2}} \circ g_{C_{\delta_1},C_{\delta_3}} = g_{C_{\delta_1},D_{\delta_2}}$ all have finite derivatives almost everywhere and $g_{C_{\delta_3},D_{\delta_2}}$ is absolutely continuous and so is an *N* function, by Theorem 3 of *Change of Variables Theorems*, the Chain Rule for the derivative of $g_{C_{\delta_3},D_{\delta_2}} \circ g_{C_{\delta_1},C_{\delta_3}} = g_{C_{\delta_1},D_{\delta_2}}$ holds almost everywhere on *I*. Therefore,

$$g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \left(g_{C_{\delta_3},D_{\delta_2}} \circ g_{C_{\delta_1},C_{\delta_3}}\right)'(x) = g_{C_{\delta_3},D_{\delta_2}}{}'\left(g_{C_{\delta_1},C_{\delta_3}}(x)\right) g_{C_{\delta_1},C_{\delta_3}}{}'(x)$$

almost everywhere on *I*. Hence,

$$g_{C_{\delta_1},D_{\delta_2}}{}'(x) = \frac{1-\delta_2}{1-\delta_3} \cdot \frac{1-\delta_3}{1-\delta_1} = \frac{1-\delta_2}{1-\delta_1} \text{ almost everywhere on } C_{\delta_1}.$$

(4) We compute the arc length of the graph of $g_{C_{\delta_1},D_{\delta_2}}$ by taking the limit of the arc length of the graph of the polygonal approximation $g_n$ of $g_{C_{\delta_1},D_{\delta_2}}$.

The arc length of $g_n$ is the sum of the length of the $2^n$ line segments, each of equal length,

$\sqrt{(\ell_n)^2 + (\lambda_n)^2} = \sqrt{\left(\frac{1}{2^n}\left(1-\delta_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)\right)^2 + \left(\frac{1}{2^n}\left(1-\delta_2\left(1-\left(\tfrac{1}{2}\right)^n\right)\right)\right)^2}$, over $I - G(n)$ plus the sum of the $2^{k-1}$ line segments, each of equal length,

$\sqrt{\left(\frac{\delta_1}{3^k}\right)^2 + \left(\frac{\delta_2}{2^{2k-1}}\right)^2} = \frac{1}{3^k}\sqrt{(\delta_1)^2 + 4\left(\frac{3}{4}\right)^{2k}(\delta_2)^2}$, over the open intervals in *U*(*k*), for *k* =1, 2, …., *n*.

Therefore, the arc length of the graph of $g_n$ is

$$2^n\sqrt{\left(\frac{1}{2^n}\left(1-\delta_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)\right)^2 + \left(\frac{1}{2^n}\left(1-\delta_2\left(1-\left(\tfrac{1}{2}\right)^n\right)\right)\right)^2} + \sum_{k=1}^{n}\frac{2^{k-1}}{3^k}\sqrt{(\delta_1)^2 + 4\left(\frac{3}{4}\right)^{2k}(\delta_2)^2}$$

$$= \sqrt{\left(1-\delta_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)^2 + \left(1-\delta_2\left(1-\left(\tfrac{1}{2}\right)^n\right)\right)^2} + \frac{1}{2}\sum_{k=1}^{n}\left(\tfrac{2}{3}\right)^k\sqrt{(\delta_1)^2 + 4\left(\frac{3}{4}\right)^{2k}(\delta_2)^2}.$$

Therefore, taking limit as *n* tends to infinity, the arc length of the graph of $g_{C_{\delta_1},D_{\delta_2}}$ is

$$= \sqrt{(1-\delta_1)^2 + (1-\delta_2)^2} + \frac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)^k \sqrt{(\delta_1)^2 + 4\left(\frac{3}{4}\right)^{2k}(\delta_2)^2}\ .$$

(5) The proof is similar to Proposition 18.

Let $G = [0,1] - C_{\delta_1} = \bigcup_{k=1}^{\infty} U(k)$, where $U(n) = \bigcup_{k=1}^{2^{n-1}} I\left(\frac{k-1}{2^{n-1}} + \frac{1}{2^n}\right)$, and $I(r) = I\left(\frac{k-1}{2^{n-1}} + \frac{1}{2^n}\right)$, $1 \le k \le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_1}{3^n}$, to be deleted in stage *n* in the construction of $C_{\gamma_1}$ . Let $H = [0,1] - D_{\delta_2} = \bigcup_{k=1}^{\infty} V(k)$, where $V(n) = \bigcup_{k=1}^{2^{n-1}} J\left(\frac{k-1}{2^{n-1}} + \frac{1}{2^n}\right)$, and $J(r) = I\left(\frac{k-1}{2^{n-1}} + \frac{1}{2^n}\right)$, $1 \le k \le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_2}{2^{2n-1}}$, to be deleted in stage *n* in the construction of $D_{\delta_2}$. Let $G(n) == \bigcup_{k=1}^{n} U(k)$ and $H(n) = \bigcup_{k=1}^{n} V(k)$. Then $G = \bigcup_{k=1}^{\infty} G(k)$ and $H = \bigcup_{k=1}^{\infty} H(k)$. $I - G(n) = F(n,1) \cup F(n,2) \cup \cdots \cup F(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\ell_n = \frac{1}{2^n}\left(1 - \delta_1\left(1 - \left(\tfrac{2}{3}\right)^n\right)\right)$ and $I - H(n) = K(n,1) \cup K(n,2) \cup \cdots \cup K(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\lambda_n = \frac{1}{2^n}\left(1 - \delta_2\left(1 - \left(\tfrac{1}{2}\right)^n\right)\right)$.

Now we examine the indexing of *U*(*n*) and *V*(*n*). We note that

$$\left\{\frac{k-1}{2^{n-1}} + \frac{1}{2^n} : 1 \le k \le 2^{n-1}\right\} = \left\{\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j} + \frac{1}{2^n} : \varepsilon_j = 0 \text{ or } 1\right\}.$$

If we let $J\left(\frac{k-1}{2^{n-1}} + \frac{1}{2^n}\right) = J\left(\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j} + \frac{1}{2^n}\right) = (c(r), d(r))$, where $r = \sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j} + \frac{1}{2^n}$, then

$$c(r) = \sum_{j=1}^{n-1}\left(\left(\lambda_j + \frac{\delta_2}{2^{2j-1}}\right)\varepsilon_j + \lambda_n\right).$$

Therefore,

$$\int_{U(n)} g_{C_{\delta_1}, D_{\delta_2}}(x)dx = \frac{\delta_1}{3^n} \times \sum_{r \in \left\{\frac{k-1}{2^{n-1}} + \frac{1}{2^n} : 1 \le k \le 2^{n-1}\right\}} c(r) + 2^{n-1}\left(\frac{1}{2} \times \frac{\delta_1}{3^n} \times \frac{\delta_2}{2^{2n-1}}\right)$$

$$=\frac{\delta_1}{3^n}\times\sum_{r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}}\left(\sum_{j=1}^{n-1}\left(\left(\lambda_j+\frac{\delta_2}{2^{2j-1}}\right)\varepsilon_j+\lambda_n\right)\right)+2^{n-2}\,\frac{\delta_1}{3^n}\times\frac{\delta_2}{2^{2n-1}}$$

$$=\frac{\delta_1}{3^n}\times\left(2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_2}{2^{2j-1}}+2^{n-1}\lambda_n\right)+2^{n-2}\,\frac{\delta_1}{3^n}\times\frac{\delta_2}{2^{2n-1}}\,.\quad\text{-------------------------}\quad(*3)$$

But $\frac{\delta_2}{2^{2j-1}}=\lambda_{j-1}-2\lambda_j$ for $1\le j\le n$. For $j = 1, \lambda_{j-1}=\lambda_0$ is set to be 1.

Therefore,

$$2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_2}{2^{2j-1}}+2^{n-1}\lambda_n+2^{n-2}\cdot\frac{\delta_2}{2^{2n-1}}$$

$$=2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\left(\lambda_{j-1}-2\lambda_j\right)+2^{n-1}\lambda_n+2^{n-2}\cdot\left(\lambda_{n-1}-2\lambda_n\right)$$

$$=2^{n-2}\sum_{j=1}^{n-1}\lambda_j-2^{n-1}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\lambda_0+2^{n-2}\sum_{j=1}^{n-2}\lambda_j+2^{n-2}\lambda_{n-1}=2^{n-2}\lambda_0=2^{n-2}\,.$$

It follows then from (*3) that

$$\int_{U(n)}g_{C_{\delta_1},D_{\delta_2}}(x)dx=\frac{\delta_1}{3^n}\times2^{n-2}=\frac{\delta_1}{2}\,\frac{2^{n-1}}{3^n}\,.$$

Hence,

$$\int_G g_{C_{\delta_1},D_{\delta_2}}(x)dx=\int_{I-C_{\gamma_1}}g_{C_{\delta_1},D_{\delta_2}}(x)dx=\sum_{n=1}^{\infty}\int_{U(n)}g_{C_{\delta_1},D_{\delta_2}}(x)dx$$

$$=\frac{\delta_1}{2}\sum_{n=1}^{\infty}\frac{2^{n-1}}{3^n}=\frac{\delta_1}{2}\,.$$

Therefore, as $\int_I g_{C_{\delta_1},D_{\delta_2}}(x)dx=\frac{1}{2}$, $\int_{C_{\delta_1}}g_{C_{\delta_1},D_{\delta_2}}(x)dx=\frac{1}{2}-\int_{I-C_{\delta_1}}g_{C_{\delta_1},D_{\delta_2}}(x)dx=\frac{1-\delta_1}{2}$.

**Remark.**

In the proof of part (3) that $g_{C_{\delta_1},D_{\delta_2}}{}'(x)=\left(g_{C_{\delta_3},D_{\delta_2}}\circ g_{C_{\delta_1},C_{\delta_3}}\right)'(x)=g_{C_{\delta_3},D_{\delta_2}}{}'\left(g_{C_{\delta_1},C_{\delta_3}}(x)\right)g_{C_{\delta_1},C_{\delta_3}}{}'(x)$

almost everywhere on $C_{\delta_1}$, instead of using Theorem 3 of *Change of Variables Theorems*, we can proceed as follows. Firstly, note that $g_{C_{\delta_1},D_{\delta_2}}=g_{C_{\delta_3},D_{\delta_2}}\circ g_{C_{\delta_1},C_{\delta_3}}$, $g_{C_{\delta_1},C_{\delta_3}}$ and $g_{C_{\delta_3},D_{\delta_2}}$ are all absolutely continuous and strictly increasing. By Proposition 16, $g_{C_{\delta_1},C_{\delta_3}}$ is differentiable almost everywhere on $C_{\delta_1}$ and $g_{C_{\delta_1},C_{\delta_3}}{}'(x)=\dfrac{1-\delta_3}{1-\delta_1}$ almost everywhere on $C_{\delta_1}$. Suppose $g_{C_{\delta_1},C_{\delta_3}}{}'(x)=\dfrac{1-\delta_3}{1-\delta_1}$ for *x* in *E*, $E\subseteq C_{\delta_1}$ and $m(C_{\delta_1}-E)=0$, $g_{C_{\delta_3},D_{\delta_2}}$ is differentiable almost everywhere in $C_{\delta_3}$ and $g_{C_{\delta_3},D_{\delta_2}}{}'(x)=\dfrac{1-\delta_2}{1-\delta_3}$ for all *x* in $D\subseteq C_{\delta_3}$ and $m(C_{\delta_3}-D)=0$.

$g_{C_{\delta_3},C_{\delta_1}}=\left(g_{C_{\delta_1},C_{\delta_3}}\right)^{-1}$ is an *N* function, since it is absolutely continuous. Therefore, $m\left(\left(g_{C_{\delta_1},C_{\delta_3}}\right)^{-1}(C_{\delta_3}-D)\right)=0$. Let $A=\left(g_{C_{\delta_1},C_{\delta_3}}\right)^{-1}(C_{\delta_3}-D)\subseteq C_{\delta_1}$. Hence, $m\left(A\cup\left(C_{\delta_1}-E\right)\right)=0$. Therefore, $g_{C_{\delta_1},C_{\delta_3}}$ is differentiable on $C_{\delta_1}-\left(A\cup\left(C_{\delta_1}-E\right)\right)=\left(C_{\delta_1}-A\right)\cap E$. Moreover, since $g_{C_{\delta_1},C_{\delta_3}}$ is injective,

$$g_{C_{\delta_1},C_{\delta_3}}\left(C_{\delta_1}-\left(A\cup\left(C_{\delta_1}-E\right)\right)\right)=g_{C_{\delta_1},C_{\delta_3}}\left(\left(C_{\delta_1}-A\right)\cap E\right)=g_{C_{\delta_1},C_{\delta_3}}\left(C_{\delta_1}-A\right)\cap g_{C_{\delta_1},C_{\delta_3}}\left(E\right).$$

This means that

$$g_{C_{\delta_1},C_{\delta_3}}\left(C_{\delta_1}-\left(A\cup\left(C_{\delta_1}-E\right)\right)\right)=\left(C_{\delta_3}-g_{C_{\delta_1},C_{\delta_3}}\left(A\right)\right)\cap g_{C_{\delta_1},C_{\delta_3}}\left(E\right)=D\cap g_{C_{\delta_1},C_{\delta_3}}\left(E\right).$$

Hence, $x\in C_{\delta_1}-\left(A\cup\left(C_{\delta_1}-E\right)\right)$ implies that $g_{C_{\delta_1},C_{\delta_3}}$ is differentiable *at x* and $g_{C_{\delta_3},D_{\delta_2}}$ is differentiable at $g_{C_{\delta_1},C_{\delta_3}}(x)$ and consequently,

$$g_{C_{\delta_1},D_{\delta_2}}{}'(x)=\left(g_{C_{\delta_3},D_{\delta_2}}\circ g_{C_{\delta_1},C_{\delta_3}}\right)'(x)=g_{C_{\delta_3},D_{\delta_2}}{}'\left(g_{C_{\delta_1},C_{\delta_3}}(x)\right)g_{C_{\delta_1},C_{\delta_3}}{}'(x)$$

$$=\frac{1-\delta_2}{1-\delta_3}\cdot\frac{1-\delta_3}{1-\delta_1}=\frac{1-\delta_2}{1-\delta_1}.$$

Next, we examine the inverse function of $g_{C_{\delta_1},D_{\delta_2}}$.

**Theorem 24.** $g_{D_{\delta_2},C_{\delta_1}}: I\to I$ is differentiable almost everywhere on *I*. $g_{D_{\delta_2},C_{\delta_1}}$ is differentiable on $[0,1]-D_{\delta_2}$. For *x* in $I-D_{\delta_2}$, if $x\in J(r)$ and $r=\sum_{k=1}^{n}\dfrac{b_k}{2^k}$, where $b_k=0$ or 1, $1\le k<n, b_n=1$, $n\ge 1$, then $g_{D_{\delta_2},C_{\delta_1}}{}'(x)=\dfrac{2^{2n-1}}{3^n}\dfrac{\delta_1}{\delta_2}=\dfrac{1}{2}\left(\dfrac{4}{3}\right)^n\dfrac{\delta_1}{\delta_2}$.

The function $g_{D_{\delta_2},C_{\delta_1}}$ is not Lipschitz on *I*.

(1) If $\delta_2=1$ and $0<\delta_1<1$, then $g_{D_{\delta_2},C_{\delta_1}}$ is not absolutely continuous on *I*.

(2) If $\delta_1=1$ and $0<\delta_2\le 1$ then $g_{D_{\delta_2},C_{\delta_1}}$ is absolutely continuous on [0, 1] and $g_{D_{\delta_2},C_{\delta_1}}{}'(x)=0$ almost everywhere on $D_{\delta_2}$.

(3) If $0<\delta_2<1$ and $0<\delta_1<1$, then $g_{D_{\delta_2},C_{\delta_1}}$ is absolutely continuous on *I* and $g_{D_{\delta_2},C_{\delta_1}}'(x)=\dfrac{1-\delta_1}{1-\delta_2}$ almost everywhere on $D_{\delta_2}$.

(4) The arc length of the graph of $g_{D_{\delta_2},C_{\delta_1}}$ is

$$\sqrt{(1-\delta_1)^2+(1-\delta_2)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)^k\sqrt{(\delta_1)^2+4(\delta_2)^2\left(\tfrac{3}{4}\right)^{2k}}.$$

(5) $\int_{I-D_{\delta_2}} g_{D_{\delta_2},C_{\delta_1}}(x)dx=\dfrac{\delta_2}{2}$ and $\int_{D_{\delta_2}} g_{D_{\delta_2},C_{\delta_1}}(x)dx=\dfrac{1-\delta_2}{2}$.

(6) $g_{D_{\delta_2},C_{\delta_1}}(x)+g_{D_{\delta_2},C_{\delta_1}}(1-x)=1$ for all *x* in *I*.

**Proof.**

Since $g_{D_{\delta_2},C_{\delta_1}}=\left(g_{C_{\delta_1},D_{\delta_2}}\right)^{-1}$ and $g_{C_{\delta_1},D_{\delta_2}}$ is strictly increasing and continuous, $g_{D_{\delta_2},C_{\delta_1}}$ is strictly increasing and continuous on *I*. Therefore, $g_{D_{\delta_2},C_{\delta_1}}$ is differentiable almost everywhere on [0,1] and $g_{D_{\delta_2},C_{\delta_1}}'$ is Lebesgue integrable.

Recall

$$\begin{aligned} H &= I-D_{\delta_2} \\ &=\bigcup\left\{J(r): r \text{ a dyadic rational, } r=\sum_{k=1}^{n}\frac{b_k}{2^k}, \text{where } b_k=0 \text{ or } 1,\ 1\le k<n, b_n=1,\ n\ge 1\right\}. \end{aligned}$$

Plainly, for $x\in J(r)$ and $r=\sum_{k=1}^{n}\frac{b_k}{2^k}$, where $b_k=0$ or $1$, $1\le k<n, b_n=1$, $n\ge 1$, then

$$g_{D_{\delta_2},C_{\delta_1}}'(x)=\frac{\text{length of } I(r)}{\text{length of } J(r)}=\frac{\frac{\delta_1}{3^n}}{\frac{\delta_2}{2^{2n-1}}}=\frac{2^{2n-1}}{3^n}\frac{\delta_1}{\delta_2}=\frac{1}{2}\left(\frac{4}{3}\right)^n\frac{\delta_1}{\delta_2}.$$

Note that $\left\{g_{D_{\delta_2},C_{\delta_1}}'(x): x\in I-H\right\}$ is unbounded as $\frac{1}{2}\left(\frac{4}{3}\right)^n\frac{\delta_1}{\delta_2}\to\infty$. Therefore, $g_{D_{\delta_2},C_{\delta_1}}$ is not Lipschitz on *I*, because if it were to be Lipschitz, then its derivatives would be bounded.

(1) If $\delta_2=1$ and $0<\delta_1<1$, then $m(g_{D_{\delta_2},C_{\delta_1}}(D_{\delta_2}))=m(g_{D_{\delta_2},C_{\delta_1}}(D_{\delta_2}))=m(C_{\delta_1})=1-\delta_1>0$. As $m(D_1)=0$, $g_{D_{\delta_2},C_{\delta_1}}$ cannot be a *N* function. Consequently, $g_{D_{\delta_2},C_{\delta_1}}$ is not absolutely continuous on *I*.

(2) If $\delta_1=1$, then $m(g_{D_{\delta_2},C_{\delta_1}}(D_{\delta_2}))=m(C_{\delta_1})=m(C_1)=1-1=0$. Moreover, $g_{D_{\delta_2},C_{\delta_1}}$ is differentiable on $I-D_{\delta_2}$. Therefore, by Theorem 12 part (a) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La*

*Vallée Poussin's Theorem*, $g_{D_{\delta_2},C_{\delta_1}}$ is absolutely continuous on *I*. By Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, $g_{D_{\delta_2},C_{\delta_1}}'(x)=0$ almost everywhere on $D_{\delta_2}$.

(3) Now, as $g_{D_{\delta_2},C_{\delta_1}} = \left(g_{C_{\delta_1},D_{\delta_2}}\right)^{-1}$ and $g_{C_{\delta_1},D_{\delta_2}}$ is strictly increasing and continuous, by Theorem 12 part (b) (Zarecki Theorem) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $g_{D_{\delta_2},C_{\delta_1}}$ is absolutely continuous if and only if $m\left(\left\{x\in[0,1]: g_{C_{\delta_1},D_{\delta_2}}'(x)=0\right\}\right)=0$.

Note that $g_{C_{\delta_1},D_{\delta_2}}'(x)>0$ for all *x* in $I-C_{\delta_1}$ and as $0<\delta_2<1$ and $0<\delta_1<1$, by Theorem 22 part (3), $g_{C_{\delta_1},D_{\delta_2}}'(x)=\dfrac{1-\delta_2}{1-\delta_1}$ almost everywhere on $C_{\delta_1}$. That means there is a subset $E\subseteq C_{\delta_1}$ such that $g_{C_{\delta_1},D_{\delta_2}}'(x)=\dfrac{1-\delta_2}{1-\delta_1}>0$ for all *x* in *E*, and $m(C_{\delta_1}-E)=0$. Hence, $\left\{x\in[0,1]: g_{C_{\delta_1},D_{\delta_2}}'(x)=0\right\}\subseteq C_{\delta_1}-E$. It follows that $m\left(\left\{x\in[0,1]: g_{C_{\delta_1},D_{\delta_2}}'(x)=0\right\}\right)=0$.
Therefore, $g_{D_{\delta_2},C_{\delta_1}}$ is absolutely continuous on *I*.

In particular, $g_{D_{\delta_2},C_{\delta_1}}$ is differentiable on $g_{C_{\delta_1},D_{\delta_2}}(E)$ and for *y* in $g_{C_{\delta_1},D_{\delta_2}}(E)$,

$$g_{D_{\delta_2},C_{\delta_1}}'(y)=\frac{1}{g_{C_{\delta_1},D_{\delta_2}}'(\left(g_{C_{\delta_1},D_{\delta_2}}\right)^{-1}(y))}=\frac{1}{g_{C_{\delta_1},D_{\delta_2}}'(g_{D_{\delta_2},C_{\delta_1}}(y))}=\frac{1}{\frac{1-\delta_2}{1-\delta_1}}=\frac{1-\delta_1}{1-\delta_2}.$$

Now since $g_{C_{\delta_1},D_{\delta_2}}$ is strictly increasing and so is injective,

$$D_{\delta_2}-g_{C_{\delta_1},D_{\delta_2}}(E)=g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1})-g_{C_{\delta_1},D_{\delta_2}}(E)=g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1}-E).$$

As $g_{C_{\delta_1},D_{\delta_2}}$ is absolutely continuous and so is a *N* function,

$$m\left(D_{\delta_2}-g_{C_{\delta_1},D_{\delta_2}}(E)\right)=m\left(g_{C_{\delta_1},D_{\delta_2}}(C_{\delta_1}-E)\right)=0.$$

It follows that $g_{D_{\delta_2},C_{\delta_1}}'(x)=\dfrac{1-\delta_1}{1-\delta_2}$ almost everywhere on $D_{\delta_2}$.

(4) The arc length of the graph of $g_{D_{\delta_2},C_{\delta_1}}$ is the same as the arc length of the graph of its inverse, $g_{C_{\delta_1},D_{\delta_2}}$ and by Theorem 22 part (4) is equal to

$$\sqrt{\left(1-\delta_1\right)^2+\left(1-\delta_2\right)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{2}{3}\right)^k\sqrt{\left(\delta_1\right)^2+4\left(\delta_2\right)^2\left(\tfrac{3}{4}\right)^{2k}}.$$

(5) The proof is similar to that of part (5) of Theorem 22.

Let $G=[0,1]-C_{\delta_1}=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}}I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $I(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_1}{3^n}$, to be deleted in stage *n* in the construction of $C_{\gamma_1}$ . Let $H=[0,1]-D_{\delta_2}=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}}J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $J(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_2}{2^{2n-1}}$, to be deleted in stage *n* in the construction of $D_{\delta_2}$. Let $G(n)==\bigcup_{k=1}^{n}U(k)$ and $H(n)=\bigcup_{k=1}^{n}V(k)$. Then $G=\bigcup_{k=1}^{\infty}G(k)$ and $H=\bigcup_{k=1}^{\infty}H(k)$. $I-G(n)=F(n,1)\cup F(n,2)\cup\cdots\cup F(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\ell_n=\frac{1}{2^n}\left(1-\delta_1\left(1-\left(\tfrac{2}{3}\right)^n\right)\right)$ and $I-H(n)=K(n,1)\cup K(n,2)\cup\cdots\cup K(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\lambda_n=\frac{1}{2^n}\left(1-\delta_2\left(1-\left(\tfrac{1}{2}\right)^n\right)\right)$.

Now we examine the indexing of the open intervals in *U*(*n*) and *V*(*n*). We note that

$$\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}=\left\{\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}:\varepsilon_j=0\text{ or }1\right\}.$$

If we let $I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)=I\left(\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}\right)=(a(r),b(r))$, where $r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}$, then

$$a(r)=\sum_{j=1}^{n-1}\left(\left(\ell_j+\frac{\delta_1}{3^j}\right)\varepsilon_j+\ell_n\right).$$

Therefore,

$$\int_{V(n)}g_{D_{\delta_2},C_{\delta_1}}(x)dx=\frac{\delta_2}{2^{2n-1}}\times\sum_{r\in\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}}a(r)+2^{n-1}\left(\frac{1}{2}\times\frac{\delta_2}{2^{2n-1}}\times\frac{\delta_1}{3^n}\right)$$

$$=\frac{\delta_2}{2^{2n-1}}\times\sum_{r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}}\left(\sum_{j=1}^{n-1}\left(\left(\ell_j+\frac{\delta_1}{3^j}\right)\varepsilon_j+\ell_n\right)\right)+2^{n-2}\frac{\delta_2}{2^{2n-1}}\times\frac{\delta_1}{3^n}$$

$$=\frac{\delta_2}{2^{2n-1}}\times\left(2^{n-2}\sum_{j=1}^{n-1}\ell_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_1}{3^j}+2^{n-1}\ell_n\right)+2^{n-2}\frac{\delta_2}{2^{2n-1}}\times\frac{\delta_1}{3^n}. \quad \text{-------------------------} \quad (*4)$$

But $\frac{\delta_1}{3^j}=\ell_{j-1}-2\ell_j$ for $1\le j\le n$. For $j = 1$, $\ell_{j-1}=\ell_0$ is set to be 1.

Therefore,

$$2^{n-2}\sum_{j=1}^{n-1}\ell_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_1}{3^j}+2^{n-1}\ell_n+2^{n-2}\cdot\frac{\delta_1}{3^n}$$

$$=2^{n-2}\sum_{j=1}^{n-1}\ell_j+2^{n-2}\sum_{j=1}^{n-1}\left(\ell_{j-1}-2\ell_j\right)+2^{n-1}\ell_n+2^{n-2}\cdot\left(\ell_{n-1}-2\ell_n\right)$$

$$=2^{n-2}\sum_{j=1}^{n-1}\ell_j-2^{n-1}\sum_{j=1}^{n-1}\ell_j+2^{n-2}\ell_0+2^{n-2}\sum_{j=1}^{n-2}\ell_j+2^{n-2}\ell_{n-1}=2^{n-2}\ell_0=2^{n-2}.$$

It follows then from (*4) that

$$\int_{V(n)}g_{D_{\delta_2},C_{\delta_1}}(x)dx=\frac{\delta_2}{2^{2n-1}}\times 2^{n-2}=\frac{\delta_2}{2}\frac{1}{2^n}.$$

Hence,

$$\int_H g_{D_{\delta_2},C_{\delta_1}}(x)dx=\int_{I-D_{\delta_2}}g_{D_{\delta_2},C_{\delta_1}}(x)dx=\sum_{n=1}^{\infty}\int_{V(n)}g_{D_{\delta_2},C_{\delta_1}}(x)dx$$

$$=\frac{\delta_2}{2}\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{\delta_2}{2}.$$

Therefore, as $\int_I g_{D_{\delta_2},C_{\delta_1}}(x)dx=\frac{1}{2}$, $\int_{D_{\delta_2}}g_{D_{\delta_2},C_{\delta_1}}(x)dx=\frac{1}{2}-\int_{I-D_{\delta_2}}g_{D_{\delta_2},C_{\delta_1}}(x)dx=\frac{1-\delta_2}{2}$.

(6) The proof is similar to Proposition 13.

Now it is ripe to introduce the general family of Cantor sets.

Let $\alpha\ge 3$. For $0<\gamma\le\alpha-2$, let $C_\gamma^\alpha$ be the Cantor set defined as follows.

In the first stage, we delete the middle open interval with length $\frac{\gamma}{\alpha}$ from [0, 1]. Following the construction for $C_\delta$, we denote this open interval by $I(1,1)$. Then the complement of this middle interval is 2 disjoint closed interval each of length $\frac{1}{2}\left(1-\frac{\gamma}{\alpha}\right)$. We denote the open deleted interval by $I(1,1)=(a(1,1),b(1,1))$. As for the case of $C_\gamma$, we denote the closed interval in the complement by $J(1,1)$ and $J(1,2)$, where the closed interval $J(1,2)$ is to the right $J(1,1)$ , meaning it is ordered in such a way that every point of $J(1,2)$ is bigger than any point in $J(1,1)$.

Then at the second stage we delete the middle open interval of length $\frac{\gamma}{\alpha^2}$ from each of the 2 remaining closed intervals. Thus, there are 2 open intervals to be deleted and they are $I(2,1)=(a(2,1),b(2,1))$ and $I(2,2)=(a(2,2),b(2,2))$. As for the case of $C_\gamma$, these two open intervals are ordered by the second index. Hence, we are left with 4 = $2^2$ remaining closed intervals, $J(2,1)$, $J(2,2)$, $J(2,3)$ and $J(2,2^2)$ each of length $\frac{1}{2^2}\left(1-\frac{\gamma}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^2\right)\right)$. Let $U(1)=I(1,1)$, $U(2)=I(2,1)\cup I(2,2)$, $G(1)=U(1)$, $G(2)=U(1)\cup U(2)$. Then $I-G(1)=J(1,1)\cup J(1,2)$, $I-G(2)=J(2,1)\cup J(2,2)\cup J(2,3)\cup J(2,2^2)$.

At stage $n$ delete the middle open interval of length $\frac{\gamma}{\alpha^n}$ from each of the $2^{n-1}$ remaining closed intervals, $J(n-1,1),J(n-1,2),\cdots,J(n-1,2^{n-1})$, each of length $\frac{1}{2^{n-1}}\left(1-\frac{\gamma}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^{n-1}\right)\right)$. Denote these open intervals by $I(n,1),I(n,2),\cdots,I(n,2^{n-1})$. Then this gives the remaining $2^n$ closed intervals, $J(n,1),J(n,2),\cdots,J(n,2^n)$, each of length $\ell_n=\frac{1}{2^n}\left(1-\frac{\gamma}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^{n}\right)\right)$. Let $U(n)=\bigcup_{k=1}^{2^{n-1}}I(n,k)$ and $G(n)=\bigcup_{k=1}^{n}U(k)=\bigcup_{k=1}^{n}\bigcup_{j=1}^{2^{k-1}}I(k,j)$.

Observe that $I-G(n)=J(n,1)\cup J(n,2)\cup\cdots\cup J(n,2^n)$.

Note that $G(n)$ consists of $2^n-1$ disjoint open intervals. The total length of the intervals in $G(n)$ is given by

$$\frac{\gamma}{\alpha}+2\frac{\gamma}{\alpha^2}+2^2\frac{\gamma}{\alpha^3}+\cdots+2^{n-1}\frac{\gamma}{\alpha^n}=\frac{\gamma}{\alpha}\left(1+\frac{2}{\alpha}+\left(\frac{2}{\alpha}\right)^2\cdots+\left(\frac{2}{\alpha}\right)^{n-1}\right)$$
$$=\frac{\gamma}{\alpha}\left(\frac{1-\left(\frac{2}{\alpha}\right)^n}{1-\frac{2}{\alpha}}\right)=\frac{\gamma}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right).$$

Note that $G(n)\subseteq G(n+1)$ and $G(n+1)=G(n)\cup U(n+1)$.

Let $G=\bigcup_{k=1}^{\infty}G(k)=\bigcup_{k=1}^{\infty}U(k)$. Thus, the measure of $G$ is the total length of all the $U(k)$, that is,

$$m(G)=\lim_{n\to\infty}=\frac{\gamma}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)=\frac{\gamma}{\alpha-2}.$$

The Cantor set $C_\gamma^\alpha$ is defined by $C_\gamma^\alpha=I-G$.

Hence, the measure of $C_\gamma^\alpha$ is given by $m(C_\gamma^\alpha) = m(I) - m(G) = 1 - \dfrac{\gamma}{\alpha-2} = \dfrac{\alpha-2-\gamma}{\alpha-2} \geq 0$.

Note that $m\left(C_{\alpha-2}^\alpha\right) = 0$.

We re-index $I(n,k_n)$, $1 \leq k_n \leq 2^{n-1}$, $1 \leq n < \infty$ as for the deleted open intervals for $C_\gamma$.

We define the Cantor function $f_{C_\gamma^\alpha}$ associated with $C_\gamma^\alpha$ in exactly the same manner as for $C_\gamma$.

For $x \in G$, $x \in I(r)$, $f_{C_\gamma^\alpha}(x) = r$.

Note that $C_\gamma^3$ is just $C_\gamma$ in Theorem 1.

All the properties that we have proved for $C_\gamma$ apply to $C_\gamma^\alpha$ and its associated Cantor Lebesgue function, $f_{C_\gamma^\alpha}$.

In summary, we have the following theorems.

**Theorem 25.** The Cantor set $C_\gamma^\alpha$ ($0 < \gamma \leq \alpha - 2$, $\alpha \geq 3$) is

(1) compact,
(2) nowhere dense, i.e., it contains no open intervals,
(3) its own boundary points,
(4) perfect, i.e., it is its own set of accumulation points,
(5) totally disconnected and
(6) between any two points in $C_\gamma^\alpha$, there is an open interval not contained in $C_\gamma^\alpha$.

**Theorem 26.** The Cantor Lebesgue function, $f_{C_\gamma^\alpha}$, associated with the Cantor set $C_\gamma^\alpha$ ($0 < \gamma \leq \alpha - 2$, $\alpha \geq 3$) is increasing and continuous and maps $C_\gamma^\alpha$ onto $I = [0,1]$.

(1) If $0 < \gamma < \alpha - 2$, then the associated Cantor Lebesgue function $f_{C_\gamma^\alpha}$ is Lipschitz with constant $\dfrac{\alpha-2}{\alpha-2-\gamma}$ and so is absolutely continuous on [0, 1]; if $\gamma = \alpha - 2$, then $f_{C_\gamma^\alpha}$ is singular and therefore not absolutely continuous.

(2) $f_{C_\gamma^\alpha}$ satisfies the relation $f_{C_\gamma^\alpha}(x) + f_{C_\gamma^\alpha}(1-x) = 1$ for all $x$ in $I$.

(3) The arc length of the graph of the Cantor Lebesgue function, $f_{C_\gamma^\alpha}:[0,1]\to[0,1]$, for $C_\gamma^\alpha$, is $\frac{\gamma}{\alpha-2}+\sqrt{1+(1-\frac{\gamma}{\alpha-2})^2}$ .

(4) $f_{C_\gamma^\alpha}$ is differentiable almost everywhere on [0,1]; $\left(f_{C_\gamma^\alpha}\right)'(x)=0$ for $x\in[0,1]-C_\gamma^\alpha$; if $0<\gamma<\alpha-2$, $\left(f_{C_\gamma^\alpha}\right)'(x)=\dfrac{\alpha-2}{\alpha-2-\gamma}$ almost everywhere on $C_\gamma^\alpha$.

(5) $\int_0^1 f_{C_\gamma^\alpha}(x)dx=\dfrac{1}{2}$, $\int_{I-C_\gamma^\alpha} f_{C_\gamma^\alpha}(x)dx=\dfrac{\gamma}{2(\alpha-2)}$ and $\int_{C_\gamma^\alpha} f_{C_\gamma^\alpha}(x)dx=\dfrac{\alpha-2-\gamma}{2(\alpha-2)}$.

The proofs for Theorem 25 and 26 are exactly the same for the Cantor set $C_\gamma$ and its associated Cantor Lebesgue function.

Let $g_{C_{\delta_1}^\alpha,C_{\delta_2}^\beta}$ be the canonical Cantor like function defined similarly as $g_{1,2}$ mapping the cantor set $C_{\delta_1}^\alpha$ onto $C_{\delta_2}^\beta$, where $0<\delta_1\le\alpha-2$ , $0<\delta_2\le\beta-2$ , and $\beta\ge 3$.

**Proposition 27.** The function, $g_{C_{\delta_1}^\alpha,C_{\delta_2}^\beta}:I\to I$ , as defined above is strictly increasing and continuous and maps the Cantor set $C_{\delta_1}^\alpha$ bijectively onto $C_{\delta_2}^\beta$, where $0<\delta_1\le\alpha-2$ and $0<\delta_2\le\beta-2$. For all *x* in *I*, $g_{C_{\delta_1}^\alpha,C_{\delta_2}^\beta}(x)+g_{C_{\delta_1}^\alpha,C_{\delta_2}^\beta}(1-x)=1$.

**Proof.** The proof is exactly the same as Proposition 15 and for the proof of the last statement, it is similar to that of Proposition 13.

Let $G=I-C_{\delta_1}^\alpha$

$$=\bigcup\left\{I(r): r \text{ a dyadic ratonal, } r=\sum_{k=1}^{n}\frac{b_k}{2^k}, \text{where } b_k=0 \text{ or } 1,\ 1\le k<n, b_n=1,\ n\ge 1\right\},$$

where $I(r)$ are the open intervals in *G* to be deleted to construct $C_{\delta_1}^\alpha$.

Let $H=I-C_{\delta_2}^\beta$

$$=\bigcup\left\{J(r): r \text{ a dyadic rational, } r=\sum_{k=1}^{n}\frac{b_k}{2^k}, \text{where } b_k=0 \text{ or } 1,\ 1\le k<n, b_n=1,\ n\ge 1\right\},$$

where $J(r)$ are the open intervals in *H* to be deleted to construct $C_{\delta_2}^\beta$.

Note that if $r=\sum_{k=1}^{n}\frac{b_k}{2^k}$, where $b_k=0$ or 1, $1\le k<n, b_n=1$, $n\ge 1$, then the length of *I*(*r*) is $\dfrac{\delta_1}{\alpha^n}$ and that of *J*(*r*) is $\dfrac{\delta_2}{\beta^n}$.

**Theorem 28.** $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}: I \to I$ is differentiable almost everywhere on *I*. $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is differentiable on $[0,1]-C_{\delta_1}^{\alpha}$. For *x* in $I-C_{\delta_1}^{\alpha}$, if $x \in I(r)$ and

$r=\sum_{k=1}^{n}\frac{b_k}{2^k}$, where $b_k=0$ or 1, $1\le k<n, b_n=1$, $n\ge 1$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{\alpha^n}{\beta^n}\frac{\delta_2}{\delta_1}=\left(\frac{\alpha}{\beta}\right)^n\frac{\delta_2}{\delta_1}$.

If $\alpha>\beta$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is not Lipschitz on *I*.

(1) If $\delta_1=\alpha-2$ and $0<\delta_2<\beta-2$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is not absolutely continuous on *I*.

(2) If $\delta_2=\beta-2$ and $0<\delta_1\le\alpha-2$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous on [0, 1] and $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=0$ almost everywhere on $C_{\delta_1}^{\alpha}$.

(3) If $0<\delta_1<\alpha-2$, $0<\delta_2\le\beta-2$ and $\alpha\le\beta$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is Lipschitz on *I* and so is absolutely continuous on *I* and $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$ almost everywhere on $C_{\delta_1}^{\alpha}$.

(4) If $0<\delta_1<\alpha-2$, $0<\delta_2<\beta-2$ and $\alpha>\beta$, then $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous on *I* but not Lipschitz on *I* and $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$ almost everywhere on $C_{\delta_1}^{\alpha}$.

(5) The arc length of the graph of $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is

$$\sqrt{\left(1-\tfrac{\delta_1}{\alpha-2}\right)^2+\left(1-\tfrac{\delta_2}{\beta-2}\right)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{2}{\alpha}\right)^k\sqrt{(\delta_1)^2+\left(\tfrac{\alpha}{\beta}\right)^{2k}(\delta_2)^2}\text{, if } \alpha\le\beta\text{, or,}$$

$$\sqrt{\left(1-\tfrac{\delta_1}{\alpha-2}\right)^2+\left(1-\tfrac{\delta_2}{\beta-2}\right)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\tfrac{2}{\beta}\right)^k\sqrt{(\delta_2)^2+\left(\tfrac{\beta}{\alpha}\right)^{2k}(\delta_1)^2}\text{, if } \alpha>\beta.$$

(6) $\int_{I-C_{\delta_1}^{\alpha}} g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}(x)dx=\frac{\delta_1}{2}\frac{1}{\alpha-2}$ and $\int_{C_{\delta_1}^{\alpha}} g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}(x)dx=\frac{1}{2}\left(1-\frac{\delta_1}{\alpha-2}\right)$.

**Proof.** Since $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is strictly increasing, it is differentiable almost everywhere on [0,1].

Plainly, if *x* is in $I-C_{\delta_1}^{\alpha}$, then $x\in I(r)$ for some

$$r=\sum_{k=1}^{n}\frac{b_k}{2^k}\text{, where } b_k=0 \text{ or } 1,\ 1\le k<n, b_n=1,\ n\ge 1$$

and
$$g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{\text{length of } J(r)}{\text{length of } I(r)}=\frac{\frac{\delta_2}{\beta^n}}{\frac{\delta_1}{\alpha^n}}=\frac{\alpha^n}{\beta^n}\frac{\delta_2}{\delta_1}=\left(\frac{\alpha}{\beta}\right)^n\frac{\delta_2}{\delta_1}.$$

If $\alpha > \beta$, then $\left(\frac{\alpha}{\beta}\right)^n \frac{\delta_2}{\delta_1} \to \infty$ as $n \to \infty$. Therefore, $\left\{ \left(g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}\right)'(x) : x \in [0,1] - C_{\delta_1}^\alpha \right\}$ is unbounded. It follows that $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is not Lipschitz on *I*.

(1) If $\delta_1 = \alpha - 2$ and $0 < \delta_2 < \beta - 2$, then
$m(g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(C_{\delta_1}^\alpha)) = m(g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(C_{\alpha-2}^\alpha)) = m(C_{\delta_2}^\beta) = \frac{\beta - 2 - \delta_2}{\beta - 2} > 0$. As $m(C_{\alpha-2}^\alpha) = 0$, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ cannot be a *N* function. Consequently, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is not absolutely continuous on *I*.

(2) If $\delta_2 = \beta - 2$, then $m(g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(C_{\delta_1}^\alpha)) = m(C_{\delta_2}^\beta) = m(C_{\beta-2}^\beta) = 0$. Moreover, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is differentiable on $I - C_{\delta_1}^\alpha$. Therefore, by Theorem 12 part (a) of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is absolutely continuous on *I*. By Theorem 15 of *Functions of Bounded Variation and Johnson's Indicatrix*, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}{}'(x) = 0$ almost everywhere on $C_{\delta_1}^\alpha$.

(3) Suppose $0 < \delta_1 < \alpha - 2$, $0 < \delta_2 \le \beta - 2$ and $\alpha \le \beta$. Let $g_n$ be the polygonal approximation of $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ determined by the points on the graph given by the end points of the open intervals in $G(n) = \bigcup_{j=1}^{n} U(j) = \bigcup_{k=1}^{2^n - 1} I\left(\frac{k}{2^n}\right)$. As the gradient of the graph of $g_n$ over each open interval in *U*(*j*) is $\left(\frac{\alpha}{\beta}\right)^j \frac{\delta_2}{\delta_1}$ and over each closed interval in $I - G(n)$ is

$$\frac{\frac{1}{2^n}\left(1 - \frac{\delta_2}{\beta-2}\left(1 - \left(\frac{2}{\beta}\right)^n\right)\right)}{\frac{1}{2^n}\left(1 - \frac{\delta_1}{\alpha-2}\left(1 - \left(\frac{2}{\alpha}\right)^n\right)\right)} = \frac{1 - \frac{\delta_2}{\beta-2}\left(1 - \left(\frac{2}{\beta}\right)^n\right)}{1 - \frac{\delta_1}{\alpha-2}\left(1 - \left(\frac{2}{\alpha}\right)^n\right)},$$

for $x \ne y$,

$$0 \le \frac{g_n(x) - g_n(y)}{x - y}$$
$$\le \max\left\{ \left(\frac{\alpha}{\beta}\right)\frac{\delta_2}{\delta_1}, \left(\frac{\alpha}{\beta}\right)^2 \frac{\delta_2}{\delta_1}, \cdots, \left(\frac{\alpha}{\beta}\right)^n \frac{\delta_2}{\delta_1}, \frac{1 - \frac{\delta_2}{\beta-2}\left(1 - \left(\frac{2}{\beta}\right)^n\right)}{1 - \frac{\delta_1}{\alpha-2}\left(1 - \left(\frac{2}{\alpha}\right)^n\right)} \right\} \le \max\left\{ \frac{\delta_2}{\delta_1}, \frac{1 - \frac{\delta_2}{\beta-2}\left(1 - \left(\frac{2}{\beta}\right)^n\right)}{1 - \frac{\delta_1}{\alpha-2}\left(1 - \left(\frac{2}{\alpha}\right)^n\right)} \right\}.$$

Therefore, taking limit as *n* tends to infinity we have, for $x \ne y$,

$$0 \le \frac{g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(x) - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(y)}{x-y} \le \max\left\{\frac{\delta_2}{\delta_1}, \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}\right\}. \text{ ------------------- (16)}$$

It follows that $\left|g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(x) - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(y)\right| \le \max\left\{\frac{\delta_2}{\delta_1}, \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}\right\}|x-y|$ for all *x*, *y* in *I*.

Hence, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is Lipschitz and so is absolutely continuous on *I*.

The case when $\delta_2 = \beta - 2$ has already been dealt with in part (2).

Now we assume $0 < \delta_1 < \alpha - 2$ and $0 < \delta_2 < \beta - 2$. Suppose $\frac{\delta_2}{\delta_1} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$.

It follows from (16) that for $x \ne y$,

$$\frac{g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(x) - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(y)}{x-y} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}.$$

Therefore, if $\frac{\delta_2}{\delta_1} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, $\left|g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(x) - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(y)\right| \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}|x-y|$ for any *x* and *y* in *I*.

Hence, if $\frac{\delta_2}{\delta_1} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is Lipschitz with constant $\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}} = \frac{\alpha-2}{\beta-2}\frac{\beta-2-\delta_2}{\alpha-2-\delta_1}$.

Therefore, if $\frac{\delta_2}{\delta_1} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, since $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}$ is strictly increasing and absolutely continuous, by Theorem 11, *Functions of Bounded Variation and Johnson's Indicatrix*,

$$\int_{C_{\delta_1}^\alpha} g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}{}'(x)dx = m\left(g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(C_{\delta_1}^\alpha)\right) = m\left(C_{\delta_2}^\beta\right) = \frac{\beta-2-\delta_2}{\beta-2} = \int_{C_{\delta_1}^\alpha} \frac{\alpha-2}{\beta-2}\frac{\beta-2-\delta_2}{\alpha-2-\delta_1}dx.$$

It follows that if $\frac{\delta_2}{\delta_1} \le \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, $\int_{C_{\delta_1}^\alpha}\left(\frac{\alpha-2}{\beta-2}\frac{\beta-2-\delta_2}{\alpha-2-\delta_1} - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}{}'(x)\right)dx = 0$ and as $\frac{g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(x) - g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}(y)}{x-y} \le \frac{1-\delta_2}{1-\delta_1}$, $g_{C_{\delta_1}^\alpha, C_{\delta_2}^\beta}{}'(x) = \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$ almost everywhere on $C_{\delta_1}^\alpha$.

Suppose $\frac{\delta_2}{\delta_1} > \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$.

Now, $\frac{\delta_2}{\delta_1} > \frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}} \Leftrightarrow \delta_1\left(\frac{1}{\delta_2} + \frac{1}{\alpha-2} - \frac{1}{\beta-2}\right) < 1 \Leftrightarrow \delta_1\left(\frac{1}{\delta_2} + \frac{\beta-\alpha}{(\alpha-2)(\beta-2)}\right) < 1$.

As the function $h(x)=x\left(\frac{1}{\delta_2}+\frac{\beta-\alpha}{(\alpha-2)(\beta-2)}\right)$ is continuous on [0, $\alpha-2$], $h(0)=0$ and

$h(\alpha-2)=\frac{\alpha-2}{\delta_2}+\frac{\beta-\alpha}{\beta-2}>\frac{\alpha-2}{\beta-2}+\frac{\beta-\alpha}{\beta-2}=1$, if $\frac{\delta_2}{\delta_1}>\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, by the Intermediate Value Theorem, there exists, $\delta_1<\delta_3<\alpha-2$ such that $h(\delta_3)=\delta_3\left(\frac{1}{\delta_2}+\frac{\beta-\alpha}{(\alpha-2)(\beta-2)}\right)=1$.

If $\frac{\delta_2}{\delta_1}>\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$, we shall factor $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ as $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}\circ g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}$, where $\frac{\delta_2}{\delta_3}\le\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_3}{\alpha-2}}$.

Then $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous on *I* and

$$g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_3}{\alpha-2}}\text{ almost everywhere on } C_{\delta_3}^{\alpha}.$$

As $0<\delta_1<\delta_3<\alpha-2$, by an argument similar to the proof of Proposition 16 part (3), $g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}$ is Lipschitz and by using $g_{C_{\delta_3}^{\alpha},C_{\delta_1}^{\alpha}}$ the inverse of $g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}$, we get

$$g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}{}'(x)=\frac{1-\frac{\delta_3}{\alpha-2}}{1-\frac{\delta_1}{\alpha-2}}\text{ almost everywhere on } C_{\delta_1}^{\alpha}.$$

Since $g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}$, $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}$ and $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}\circ g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}=g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ all have finite derivatives almost everywhere and $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous and so is an *N* function, by Theorem 3 of *Change of Variables Theorems*, the Chain Rule for the derivative of $g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}\circ g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}=g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ holds almost everywhere on *I*. Therefore,

$$g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\left(g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}\circ g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}\right)'(x)=g_{C_{\delta_3}^{\alpha},C_{\delta_2}^{\beta}}{}'\left(g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}(x)\right)g_{C_{\delta_1}^{\alpha},C_{\delta_3}^{\alpha}}{}'(x)$$

almost everywhere on *I*.

Hence, $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_3}{\alpha-2}}\cdot\frac{1-\frac{\delta_3}{\alpha-2}}{1-\frac{\delta_1}{\alpha-2}}=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$ almost everywhere on $C_{\delta_1}^{\alpha}$.

(4) Suppose $0<\delta_1<\alpha-2$, $0<\delta_2<\beta-2$ and $\alpha>\beta$,

Now, as $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}=\left(g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}\right)^{-1}$ and $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}$ is strictly increasing and continuous, by Theorem 12 part (b) (Zarecki Theorem) of *Functions Having Finite Derivatives, Bounded Variation,* *Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous if and only if $m\left(\left\{x\in[0,1]: g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)=0\right\}\right)=0$.

Note that by part (3), $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)>0$ for all *x* in $I-C_{\delta_2}^{\beta}$ and as $0<\delta_2<\beta-2$ and $0<\delta_1<\alpha-2$, by part (3), $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)=\dfrac{1-\frac{\delta_1}{\alpha-2}}{1-\frac{\delta_2}{\beta-2}}$ almost everywhere on $C_{\delta_2}^{\beta}$. That means there is a subset $E\subseteq C_{\delta_2}^{\beta}$ such that $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)=\dfrac{1-\frac{\delta_1}{\alpha-2}}{1-\frac{\delta_2}{\beta-2}}>0$ for all *x* in *E*, and $m(C_{\delta_2}^{\beta}-E)=0$.

Hence, $\left\{x\in[0,1]: g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)=0\right\}\subseteq C_{\delta_2}^{\beta}-E$. It follows that $m\left(\left\{x\in[0,1]: g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(x)=0\right\}\right)=0$. Therefore, $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is absolutely continuous on *I*.

In particular, $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is differentiable on $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(E)$ and for *y* in $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(E)$,

$$g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(y)=\frac{1}{g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'((g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}})^{-1}(y))}=\frac{1}{g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}{}'(g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}(y))}=\frac{1}{\dfrac{1-\frac{\delta_1}{\alpha-2}}{1-\frac{\delta_2}{\beta-2}}}=\frac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}.$$

Now since $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}$ is strictly increasing and so is injective,

$$C_{\delta_1}^{\alpha}-g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(E)=g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(C_{\delta_2}^{\beta})-g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(E)=g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(C_{\delta_2}^{\beta}-E).$$

As $g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}$ is absolutely continuous and so is a *N* function,

$$m\left(C_{\delta_1}^{\alpha}-g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(E)\right)=m\left(g_{C_{\delta_2}^{\beta},C_{\delta_1}^{\alpha}}(C_{\delta_2}^{\beta}-E)\right)=0.$$

It follows that $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}{}'(x)=\dfrac{1-\frac{\delta_2}{\beta-2}}{1-\frac{\delta_1}{\alpha-2}}$ almost everywhere on $C_{\delta_1}^{\alpha}$.

(5) We compute the arc length of the graph of $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ by taking the limit of the arc length of the graph of the polygonal approximation $g_n$ of $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$.

The arc length of $g_n$ is the sum of the length of the $2^n$ line segments, each of equal length,

$\sqrt{(\ell_n)^2+(\lambda_n)^2}=\sqrt{\left(\frac{1}{2^n}\left(1-\frac{\delta_1}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)\right)\right)^2+\left(\frac{1}{2^n}\left(1-\frac{\delta_2}{\beta-2}\left(1-\left(\frac{2}{\beta}\right)^n\right)\right)\right)^2}$, over $I-G(n)$ plus the sum of the $2^{k-1}$ line segments, each of equal length, $\sqrt{\left(\frac{\delta_1}{\alpha^k}\right)^2+\left(\frac{\delta_2}{\beta^k}\right)^2}$, over the open intervals in $U(k)$, for $k$ =1, 2, …., $n$.

Therefore, the arc length of the graph of $g_n$ is

$$2^n\sqrt{\left(\frac{1}{2^n}\left(1-\frac{\delta_1}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)\right)\right)^2+\left(\frac{1}{2^n}\left(1-\frac{\delta_2}{\beta-2}\left(1-\left(\frac{2}{\beta}\right)^n\right)\right)\right)^2}+\sum_{k=1}^{n}2^{k-1}\sqrt{\left(\frac{\delta_1}{\alpha^k}\right)^2+\left(\frac{\delta_2}{\beta^k}\right)^2}$$

$$=\sqrt{\left(1-\frac{\delta_1}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)\right)^2+\left(1-\frac{\delta_2}{\beta-2}\left(1-\left(\frac{2}{\beta}\right)^n\right)\right)^2}+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{2}{\alpha}\right)^k\sqrt{(\delta_1)^2+\left(\frac{\alpha}{\beta}\right)^{2k}(\delta_2)^2},$$

if $\alpha\le\beta$,

or

$$=\sqrt{\left(1-\frac{\delta_1}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)\right)^2+\left(1-\frac{\delta_2}{\beta-2}\left(1-\left(\frac{2}{\beta}\right)^n\right)\right)^2}+\frac{1}{2}\sum_{k=1}^{n}\left(\frac{2}{\beta}\right)^k\sqrt{(\delta_2)^2+\left(\frac{\beta}{\alpha}\right)^{2k}(\delta_1)^2},$$

if $\alpha>\beta$.

Therefore, taking limit as $n$ tends to infinity, the arc length of the graph of $g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}$ is

$$\sqrt{\left(1-\frac{\delta_1}{\alpha-2}\right)^2+\left(1-\frac{\delta_2}{\beta-2}\right)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{\alpha}\right)^k\sqrt{(\delta_1)^2+\left(\frac{\alpha}{\beta}\right)^{2k}(\delta_2)^2}\text{, if }\alpha\le\beta\text{, or,}$$

$$\sqrt{\left(1-\frac{\delta_1}{\alpha-2}\right)^2+\left(1-\frac{\delta_2}{\beta-2}\right)^2}+\frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{2}{\beta}\right)^k\sqrt{(\delta_2)^2+\left(\frac{\beta}{\alpha}\right)^{2k}(\delta_1)^2}\text{, if }\alpha>\beta.$$

(6) The proof is similar to Proposition 18

Let $G=[0,1]-C_{\delta_1}=\bigcup_{k=1}^{\infty}U(k)$, where $U(n)=\bigcup_{k=1}^{2^{n-1}}I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $I(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_1}{3^n}$, to be deleted in stage $n$ in the construction of $C_{\gamma_1}$. Let $H=[0,1]-D_{\delta_2}=\bigcup_{k=1}^{\infty}V(k)$, where $V(n)=\bigcup_{k=1}^{2^{n-1}}J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, and $J(r)=I\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)$, $1\le k\le 2^{n-1}$, are the open intervals, each of length $\frac{\delta_2}{2^{2n-1}}$, to be deleted in stage $n$ in the construction of $D_{\delta_2}$. Let $G(n)==\bigcup_{k=1}^{n}U(k)$ and $H(n)=\bigcup_{k=1}^{n}V(k)$. Then

$G=\bigcup_{k=1}^{\infty}G(k)$ and $H=\bigcup_{k=1}^{\infty}H(k)$. $I-G(n)=F(n,1)\cup F(n,2)\cup\cdots\cup F(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\ell_n=\frac{1}{2^n}\left(1-\frac{\delta_1}{\alpha-2}\left(1-\left(\frac{2}{\alpha}\right)^n\right)\right)$ and $I-H(n)=K(n,1)\cup K(n,2)\cup\cdots\cup K(n,2^n)$ is a disjoint union of $2^n$ closed interval of length $\lambda_n=\frac{1}{2^n}\left(1-\frac{\delta_2}{\beta-2}\left(1-\left(\frac{2}{\beta}\right)^n\right)\right)$.

Now we examine the indexing of $U(n)$ and $V(n)$. We note that

$$\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}=\left\{\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}:\varepsilon_j=0\text{ or }1\right\}.$$

If we let $J\left(\frac{k-1}{2^{n-1}}+\frac{1}{2^n}\right)=J\left(\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}\right)=(c(r),d(r))$, where $r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}$, then

$$c(r)=\sum_{j=1}^{n-1}\left(\left(\lambda_j+\frac{\delta_2}{2^{2j-1}}\right)\varepsilon_j+\lambda_n\right).$$

Therefore,

$$\int_{U(n)}g_{C_{\delta_1}^{\alpha},C_{\delta_2}^{\beta}}(x)dx=\frac{\delta_1}{\alpha^n}\times\sum_{r\in\left\{\frac{k-1}{2^{n-1}}+\frac{1}{2^n}:1\le k\le 2^{n-1}\right\}}c(r)+2^{n-1}\left(\frac{1}{2}\times\frac{\delta_1}{\alpha^n}\times\frac{\delta_2}{\beta^n}\right)$$

$$=\frac{\delta_1}{\alpha^n}\times\sum_{r=\sum_{j=1}^{n-1}\frac{\varepsilon_j}{2^j}+\frac{1}{2^n}}\left(\sum_{j=1}^{n-1}\left(\left(\lambda_j+\frac{\delta_2}{2^{2j-1}}\right)\varepsilon_j+\lambda_n\right)\right)+2^{n-2}\frac{\delta_1}{\alpha^n}\times\frac{\delta_2}{\beta^n}$$

$$=\frac{\delta_1}{\alpha^n}\times\left(2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_2}{2^{2j-1}}+2^{n-1}\lambda_n\right)+2^{n-2}\frac{\delta_1}{\alpha^n}\times\frac{\delta_2}{\beta^n}.\quad\text{-------------------------}\quad(*5)$$

But $\frac{\delta_2}{\beta^j}=\lambda_{j-1}-2\lambda_j$ for $1\le j\le n$. For $j=1$, $\lambda_{j-1}=\lambda_0$ is set to be 1.

Therefore,

$$2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\frac{\delta_2}{\beta^j}+2^{n-1}\lambda_n+2^{n-2}\cdot\frac{\delta_2}{\beta^n}$$

$$=2^{n-2}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\sum_{j=1}^{n-1}\left(\lambda_{j-1}-2\lambda_j\right)+2^{n-1}\lambda_n+2^{n-2}\cdot\left(\lambda_{n-1}-2\lambda_n\right)$$

$$=2^{n-2}\sum_{j=1}^{n-1}\lambda_j-2^{n-1}\sum_{j=1}^{n-1}\lambda_j+2^{n-2}\lambda_0+2^{n-2}\sum_{j=1}^{n-2}\lambda_j+2^{n-2}\lambda_{n-1}=2^{n-2}\lambda_0=2^{n-2}.$$

It follows then from (*5) that

$$\int_{U(n)} g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \frac{\delta_1}{\alpha^n}\times 2^{n-2} = \frac{\delta_1}{2}\frac{2^{n-1}}{\alpha^n}.$$

Hence,

$$\int_G g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \int_{I-C_{\delta_1}^{\alpha}} g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \sum_{n=1}^{\infty}\int_{U(n)} g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx$$

$$= \frac{\delta_1}{2}\sum_{n=1}^{\infty}\frac{2^{n-1}}{\alpha^n} = \frac{\delta_1}{4}\frac{2}{\alpha}\frac{1}{1-\frac{2}{\alpha}} = \frac{\delta_1}{2}\frac{1}{\alpha-2}.$$

Therefore, as $\int_I g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \frac{1}{2}$,

$$\int_{C_{\delta_1}^{\alpha}} g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \frac{1}{2} - \int_{I-C_{\delta_1}^{\alpha}} g_{C_{\delta_1}^{\alpha}, C_{\delta_2}^{\beta}}(x)dx = \frac{1}{2} - \frac{\delta_1}{2}\frac{1}{\alpha-2} = \frac{1}{2}\left(1-\frac{\delta_1}{\alpha-2}\right).$$

This completes the proof of the theorem.

Now we present a characterization of a continuous monotone function with maximum arc length for its graph.

**Theorem 29.** Suppose $f:[0,1]\to[0,1]$ is a continuous increasing function with $f(0)=0$ and $f(1) = 1$. Then the graph of $f$ has maximum arc length (=2), if and only if, $f$ is singular.

**Proof.**

Note that if $f$ is singular, then it cannot be absolutely continuous. This is because if $f$ is absolutely continuous on [0, 1] and $f'(x) = 0$ almost everywhere, then by Theorem 9 of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $f$ is a constant function contradicting $f(0)=0$ and $f(1) = 1$.

Thus, if $f$ is singular, then as $f$ is monotone increasing and continuous, the arc length of the graph of $f$ is given by,

$$\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx + T_h[0,1],$$

where $h(x) = f(x) - \int_0^x f'(x)dx$ is an increasing continuous function and is the singular part of $f$ in the Lebesgue decomposition of $f$ and $T_h[0,1]$ is the total variation of $h$ on [0,1]. (See Theorem 9 of *Arc Length, Functions of Bounded Variation and Total Variation*.) Moreover, $T_h[0,1] = h(1) - h(0) = f(1) - (0) = 1$.

Hence the arc length of the graph of $f$ is given by $\int_{[0,1]} 1dx + 1 = 2$. Note that the maximum arc length of a graph of monotone function from [0, 1] to [0, 1] is 2.

Conversely, suppose the arc length of the graph of $f$ is 2.

If $f$ is not absolutely continuous on [0, 1], then the arc length of its graph is given by

$$\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx+T_h[0,1]=\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx+f(1)-\int_{[0,1]}f'(x)dx\,.$$

Thus, $2=\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx+1-\int_{[0,1]}f'(x)dx$.

Consequently, $\int_{[0,1]}\left(1+f'(x)\right)dx=\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx$.

As $1+f'(x)\geq\sqrt{1+\left(f'(x)\right)^2}$ almost everywhere on [0, 1], it follows that $1+f'(x)=\sqrt{1+\left(f'(x)\right)^2}$ almost everywhere on [0, 1]. By squaring both sides, we deduce immediately that $f'(x)=0$ almost everywhere on [0, 1].

Therefore, $f$ is singular.

If $f$ is absolutely continuous on [0, 1], then the arc length of its graph is given by

$$\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx\,.$$

Therefore,

$$2=\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx\leq\int_{[0,1]}\left(1+f'(x)\right)dx=1+\int_{[0,1]}f'(x)dx=1+m(f([0,1])=1+1=2\,,$$

since $\int_{[0,1]}f'(x)dx=m\left(f([0,1])\right)$ by Theorem 11, *Functions of Bounded Variation and Johnson's Indicatrix*, because $f$ is a monotone increasing and absolutely continuous function.

It follows that $\int_{[0,1]}\sqrt{1+\left(f'(x)\right)^2}\,dx=\int_{[0,1]}\left(1+f'(x)\right)dx$. As $1+f'(x)\geq\sqrt{1+\left(f'(x)\right)^2}$ almost everywhere on [0, 1], it follows that $1+f'(x)=\sqrt{1+\left(f'(x)\right)^2}$ almost everywhere on [0, 1].

By squaring both sides, we conclude immediately that $f'(x)=0$ almost everywhere on [0, 1].

As $f$ is absolutely continuous, $f$ is a constant function, contradicting that $f(0)=0$ and $f(1)=1$.

Hence, if the arc length of the graph of $f$ is 2, $f$ cannot be absolutely continuous and must be singular.

This completes the proof.

**Remark.**

An example of an increasing continuous function $f:[0,1]\to\mathbb{R}$ with $f(0)=1$ and $f(1)=1$ and having maximum arc length for its graph is the ternary Cantor function, $f_{C_1}$.

We can easily translate Theorem 29 to the more general case as stated below. The proof is exactly the same.

**Theorem 30.** Suppose $f:[a,b]\to[c,d]$ is a continuous increasing function with *f* (*a*) = *c* and *f* (*b*) = *d*. Then the graph of *f* has maximum arc length equal to *b* − *a* + *d* − *c*, if and only if, *f* is singular.

**Remark.**

1. Let $f=h\circ f_{C_1}\circ g$, where $f_{C_1}$ is the Cantor function for the ternary Cantor set, $g(x)=\dfrac{1}{b-a}(x-a)$ and $h(y)=(d-c)y+c$. Note that *f* is monotone increasing and continuous on [*a*, *b*]. The function $f_{C_1}$ is singular and so $f_{C_1}{}'(x)=0$ almost everywhere on [0, 1]. Let $E=\left\{x\in[0,1]:f_{C_1}{}'(x)=0\right\}$. Then $m(E)=1$ and $m([0,1]-E)=0$. Note that $g^{-1}(x)=(b-a)x+a$ is obviously a *N* function and so $m\left(g^{-1}([0,1]-E)\right)=0$. Since $g^{-1}$ is a continuous strictly increasing function and $g^{-1}([0,1])=[a,b]$, $g^{-1}([0,1]-E)=[a,b]-g^{-1}(E)$.

Indeed, we may take $E=[0,1]-C_1$. Then $E=G=\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{2^{k-1}}I(k,j)$, where *G*, as described in the initial introduction section on the Cantor set $C_\gamma$, is the disjoint union of open intervals to be deleted in the construction of the ternary Cantor set $C_1$. Since $g^{-1}$ is a continuous linear function, $m\left(g^{-1}(E)\right)=\left(b-a\right)m(E)=(b-a)\cdot1=b-a$. It follows that $m\left([a,b]-g^{-1}(E)\right)=0$. Let $F=g^{-1}(E)$. Then by the Chain Rule, for $x\in F$,

$$f'(x)=h'\left(f_{C_1}\circ g(x)\right)f_{C_1}{}'\left(g(x)\right)g'(x)=(d-c)\cdot\frac{1}{b-a}\cdot f_{C_1}{}'\left(g(x)\right)=0.$$

Thus, as $m\left([a,b]-F\right)=0$, $f'(x)=0$ almost everywhere on $[a,b]$ and so is singular. This gives a function *f* , whose arc length is the possible maximum arc length.

2. Similar result follows from Theorem 30 for continuous monotone function. If *f* is continuous and monotone decreasing, then −*f* is continuous and monotone increasing. Applying Theorem 30 gives the result for continuous monotone decreasing function. We may state this result as follows.

**Theorem 31.** Suppose $f:[a,b]\to[c,d]$ is a continuous monotone function. Then the graph of *f* has maximum arc length equal to *b*− *a* + *d* – *c*, if and only if, *f* is singular.

Recall that the ternary Cantor function $f_{C_1}:[0,1]\to[0,1]$ is singular and that $m\left(f_{C_1}(C_1)\right)=[0,1]$ , with $m(C_1)=0$. Does this property of having a set of measure zero such that the measure of its image is equal to the length of [0, 1] implies that the continuous monotone function $f_{C_1}$ is singular? The answer is "yes". We formulate this result in a slightly more general setting in the next theorem.

**Theorem 32.** Suppose $f:[a,b]\to[c,d]$ is a continuous monotone function that maps [*a*, *b*] onto [*c*, *d*]. If there exists a subset $E\subseteq[a,b]$ such that $m(E)=0$ and $m\left(f(E)\right)=d-c$, then *f* is singular.

**Proof.** We shall prove the theorem for the case when *f* is increasing and onto (and therefore, continuous). (As *f* is increasing on [*a*, *b*], *f* can have only jump discontinuities. But as the range of *f* is an interval, no such jump discontinuity exists and so *f* is continuous on [*a*, *b*].)

Since *f* is increasing, *f* is differentiable almost everywhere on [*a*, *b*] and so there exists a subset $D\subseteq[a,b]$ such that *f* is differentiable on $[a,b]-D$ and *m*(*D*) = 0. We may assume without loss of generality that $D\subseteq E$. (If need be we may replace *E* by $E\cup D$.) Thus, *f* is differentiable on $[a,b]-E$ , $m(E)=0$ and $m\left(f(E)\right)=d-c$. Hence, $m\left([c,d]-f(E)\right)=0$.

Let $H=f^{-1}\left([c,d]-f(E)\right)$.

Let $M=\left\{x: f \text{ is differentiable at } x \text{ finitely or infinitely and } f'(x)\neq 0\right\}$. Then $m\left(f(H\cap M)\right)=0$.

By Theorem 2 of *Change of Variables Theorems*, $f'(x)=0$ almost everywhere on $H\cap M$.

If $x\in H\cap M$ , then $f'(x)\neq 0$. Therefore, $m(H\cap M)=0$. It follows that $f'(x)=0$ almost everywhere on *H*. Note that *H* may be empty.

If $H=[a,b]-E$ , for instance, when *f* is strictly increasing, then $f'(x)=0$ almost everywhere on [*a*, *b*], i.e., *f* is singular.

Suppose $f\left([a,b]-E\right)\cap f(E)=\varnothing$. Then $f\left([a,b]-E\right)=[c,d]-f(E)$ and $H=f^{-1}\left([c,d]-f(E)\right)=f^{-1}\left(f\left([a,b]-E\right)\right)=[a,b]-E$. Hence, $f'(x)=0$ almost everywhere on [*a*, *b*], i.e., *f* is singular.

Suppose $f\left([a,b]-E\right)\cap f(E)\neq\varnothing$. Let $F=f^{-1}\left(f\left([a,b]-E\right)\cap f(E)\right)$.
Now

$$[c,d]=\left(f(E)\cap f\left([a,b]-E\right)\right)\cup\left(f(E)-f(E)\cap f\left([a,b]-E\right)\right)\cup\left(f\left([a,b]-E\right)-f(E)\cap f\left([a,b]-E\right)\right)$$

is a disjoint union. Note that $f^{-1}\left(f(E)-f(E)\cap f\left([a,b]-E\right)\right)\subseteq E$ and $f\left([a,b]-E\right)-f(E)\cap f\left([a,b]-E\right)\subseteq[c,d]-f(E)$. Therefore, $m\left(f^{-1}\left(f(E)-f(E)\cap f\left([a,b]-E\right)\right)\right)=0$ and $f^{-1}\left(f\left([a,b]-E\right)-f(E)\cap f\left([a,b]-E\right)\right)\subseteq f^{-1}\left([c,d]-f(E)\right)=H$. We have already shown that $f'(x)=0$ almost everywhere on *H*. Therefore, $f'(x)=0$ almost everywhere on $f^{-1}\left(f\left([a,b]-E\right)-f(E)\cap f\left([a,b]-E\right)\right)$.

Let $y\in f\left([a,b]-E\right)\cap f(E)$. Since *f* is continuous, $f^{-1}(y)$ is closed. Now $f^{-1}(y)$ is not a singleton set as there exists $e\in E$ and $x\in[a,b]-E$ such that $f(x)=f(e)=y$. Moreover, $f^{-1}(y)$ is an interval. This is because if $\alpha$ and $\beta$ are in $f^{-1}(y)$ with $\alpha<\beta$, then since *f* is increasing and $f(\alpha)=f(\beta)=y$, $f(x)=y$ for all *x* in $[\alpha,\beta]$, the interval $[\alpha,\beta]$ is in $f^{-1}(y)$. It follows that $f^{-1}(y)$ is a closed interval and is obviously bounded. Therefore, $f'(x)=0$ for all *x* in the interior of $f^{-1}(y)$. If *f* is differentiable at the end point $\gamma$ of $f^{-1}(y)$, then $f'(\gamma)=0$. If *f* is not differentiable at the end point $\gamma$ of $f^{-1}(y)$, then $\gamma\in E$. Thus $f'(x)=0$ for *x* in $f^{-1}(y)-E$. Since this is true for each *y* in $f\left([a,b]-E\right)\cap f(E)$, $f'(x)=0$ for *x* in $F-E=f^{-1}\left(f\left([a,b]-E\right)\cap f(E)\right)-E$. Since *E* is of measure zero, $f'(x)=0$ almost everywhere on $F=f^{-1}\left(f\left([a,b]-E\right)\cap f(E)\right)$. It follows that $f'(x)=0$ almost everywhere on $[a,b]$.

This proves that *f* is singular on [*a*, *b*].

We have seen that the ternary Cantor function is an example of a monotone increasing singular functions. Does there exist a strictly increasing singular function? R. Salem, in *On some singular monotonic functions which are strictly increasing*, *Transaction American Mathematical Society, 53 (1943), 427-439*, gave an example together with the Minkowski's function, ?(*x*). In 1952, F. Riesz and Sz.-Nagy in their book, *Functional Analysis*, page 48-49, gave an example of such a strictly increasing singular function. In 1978, L. Takács, in *An increasing continuous singular function, American Mathematical Monthly 85 (1) (1978) 35-37*, gave a family of increasing continuous singular functions. In *Riesz-Nagy Singular functions revisited*, *J. Math. Anal. Appl. 329 (2007) 592–602*, Jaume Paradís , Pelegrí Viader and Lluís Bibiloni, gave a generalization of Riesz-Nagy singular functions and Takács singular functions and showed that these two families are related. Thus, strictly increasing singular functions are abundant.

**Theorem 33.** Suppose $f:[a,b]\to[c,d]$ is a continuous strictly monotone function that maps [*a*, *b*] onto [*c*, *d*]. Then *f* is singular, if and only if, the inverse function of *f*, $g=f^{-1}$ is singular.

**Proof.**

Since $f$ is continuous and strictly monotone, its inverse $g = f^{-1}$ is also continuous and strictly monotone.

By Theorem 31, $f$ is singular, if and only if, the graph of $f$ has maximum arc length equal to $b - a + d - c$, if and only if, the graph of its inverse, $g$, has maximum arc length equal to $b - a + d - c$, if and only if, $g$ is singular.

**Remark.**

1. The ternary Cantor function, $f_{C_1}$, is continuous, monotone and singular and if $f_{C_1}$ is differentiable at $x$, then $f_{C_1}{}'(x) = 0$. We have noted in Proposition 8 that $f_{C_1}{}'(x) = 0$ for $x$ in $[0,1] - C_1$. As $m(C_1) = 0$, $f_{C_1}$ is differentiable outside a set of measure zero and its derivative is zero outside $C_1$. More is true $f_{C_1}$ is not differentiable at any point of $C_1$. We deduce this as follows. If $x \in C_1$, then since $C_1 = G^c = \left(\bigcup_{k=1}^{\infty} G(k)\right)^c = \bigcap_{k=1}^{\infty} \left(G(k)\right)^c$, $x \in \left(G(k)\right)^c$ for integer $k \geq 1$. Now $\left(G(k)\right)^c = J(k,1) \cup J(k,2) \cup \cdots \cup J(k,2^k)$ is a disjoint union of closed intervals, each of length $\frac{1}{3^k}$. So, if $x \in \left(G(k)\right)^c$, $x \in J(k,n)$ for some $1 \leq n \leq 2^k$. Let $J(k,n) = [a(k,n), b(k,n)]$. Then $\dfrac{f_{C_1}(b(k,n)) - f_{C_1}(a(k,n))}{b(k,n) - a(k,n)} = \dfrac{\frac{1}{2^k}}{\frac{1}{3^k}} = \left(\dfrac{3}{2}\right)^k$. If $x = a(k,n)$ or $x = b(k,n)$, then $\dfrac{f_{C_1}(b(k,n)) - f_{C_1}(x)}{b(k,n) - x} = \left(\dfrac{3}{2}\right)^k$ or $\dfrac{f_{C_1}(x) - f_{C_1}(a(k,n))}{x - a(k,n)} = \left(\dfrac{3}{2}\right)^k$. If $a(k,n) < x < b(k,n)$, then
$\max\left\{\dfrac{f_{C_1}(b(k,n)) - f_{C_1}(x)}{b(k,n) - x}, \dfrac{f_{C_1}(x) - f_{C_1}(a(k,n))}{x - a(k,n)}\right\} \geq \dfrac{f_{C_1}(b(k,n)) - f_{C_1}(a(k,n))}{b(k,n) - a(k,n)} = \left(\dfrac{3}{2}\right)^k$, by applying the inequality $\max\left\{\dfrac{a}{b}, \dfrac{c}{d}\right\} \geq \dfrac{a+c}{b+d}$ for $a \geq 0, c \geq 0, b > 0$ and $d > 0$. Since $\left(\dfrac{3}{2}\right)^k \to \infty$ as $k \to \infty$, $f_{C_1}$ has an infinite derived number at $x$ and so $f_{C_1}$ is not differentiable at $x$. It follows that $f_{C_1}$ is not differentiable at every point in $C_1$. In summary, $f_{C_1}$ has the property: if $f_{C_1}{}'(x)$ exists and is finite, then $f_{C_1}{}'(x) = 0$. Riesz-Nagy singular function as shown in their book, *Functional Analysis,* has this property. Salem's family of singular functions also exhibits this property, see Theorem 2 of *Singular Functions with Applications to Fractal Dimensions and Generalized Takagi Functions*, *J. Acta Appl Math (2012) 119, 129-148*, by E. de Amo, M. Díaz Carrillo and J. Fernández-Sánchez. Minkowski's Question

mark function has this property, see Theorem 3.1, in *The Derivative of Minkowski's ?(x) Function, Journal of Mathematical Analysis and Applications 253, 107-125 (2001)* by *J. Paradıs and P. Viader*. Takács singular function also possesses this property. All the singular functions mentioned so far have the property that whenever the function is differentiable, the derivative has to be zero.

We have two curious questions:

(i) Does there exist a continuous increasing singular function, $f$, with $f'(x) > 0$ at some points or subset of measure zero?

(ii) Does there exist a continuous increasing singular function, $f$, such that the set $\{x : f \text{ is not differentiable at } x\}$ is not no-where dense?

Both questions have affirmative answer.

For question (i), Juan Fern´Andez S´Anchez, Pelegr´I Viader, Jaume Parad´Is and Manuel D´Iaz Carrillo, in *A Singular Function With A Non-Zero Finite Derivative On A Dense Set*, *Nonlinear Analysis: Theory, Methods & Applications, 95, (2014), 703-713*, gave an example, a function $H : [0,1] \to [0,1]$, which is singular, strictly increasing, with non-zero derivative on a dense subset of [0, 1]. For question (ii), Salem's singular function, given in *On some singular monotonic functions which are strictly increasing*, *Transaction American Mathematical Society, 53 (1943), 427-439*, is strictly increasing, continuous and singular, whose set of non-differentiability is dense in [0, 1]. Salem's construction is geometric and yields a function whose set of non-differentiability contains $\left\{ \frac{k}{2^n} \in (0,1) : k, n \text{ positive integer} \right\}$, which is obviously dense in [0, 1]. A proof of the function being singular is by showing that if *x* is in the set of *normal numbers* to the base 2, which has measure 1 and if the function is differentiable at *x*, then the derivative has to be zero. For the details, please refer to Salem's paper. The proof that the derivative can only take on zero derivative, whenever the function is differentiable is much harder.

We close the article with the following interesting observation about continuous bijective function.

**Theorem 34.** Suppose $f : [a,b] \to [c,d]$ is a continuous bijective function that maps [*a*, *b*] onto [*c*, *d*]. Then the following is equivalent,

(1) $f$ is singular.
(2) $f^{-1}$ is singular.
(3) Arc length of the graph of $f$ is $b - a + d - c$.

(4) There exists a set $E$ in $[a, b]$ such that $m(E) = b–a$ and $m\big(f(E)\big) = 0$.

(5) There exists a set $E$ in $[a, b]$ such that $m(E) = 0$ and $m\big(f(E)\big) = d - c$.

**Proof.** A continuous bijective function mapping $[a, b]$ onto $[c, d]$ is strictly monotone.

By Theorem 33, (1) $\Leftrightarrow$ (2). By Theorem 31, (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$(4).

If $f$ is singular, then $f'(x) = 0$ almost everywhere on $[a, b]$. Therefore, the set $E = \{x : f \text{ is differentiable at } x \text{ and } f'(x) = 0\}$ has measure equal to $b–a$. By Theorem 3 of *Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*, $m\big(f(E)\big) = 0$.

(4) $\Rightarrow$(1). Since $f$ is monotone, $f$ is differentiable almost everywhere on $[a, b]$. Therefore, there exists a set $F$ in $[a, b]$ such that $f$ is differentiable on $[a,b] - F$ and $m(F) = 0$. Since $m(E) = b–a$, $m(E–F) = b–a$. Moreover, $m(f(E - F)) = 0$, since $m(f(E)) = 0$. Note that $f$ is differentiable on $E–F$. Therefore, by Theorem 2 of *Change of Variables Theorems*, $f'(x) = 0$ almost everywhere on $E–F$. As $m(E–F) = b–a$, $f'(x) = 0$ almost everywhere on $[a, b]$.

(5) $\Rightarrow$(1). This is just Theorem 32.

(1) $\Rightarrow$(5). Suppose $f$ is singular. Then by (2), $f^{-1}$ is singular. Then by (4), There exists a set $F$ in $[c, d]$ such that $m(F) = d–c$ and $m\big(f^{-1}(F)\big) = 0$. Let $E = f^{-1}(F)$. Then $m\big(E\big) = 0$ and $m(f(E)) = m(F) = d - c$.