National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 9

1. By using the binomial series expansion for $\frac{1}{\sqrt{1-x^2}}$, show that

$$\sin^{-1}(x) = x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{x^{2n+1}}{2n+1}, \text{ for } |x| < 1$$

2. Using question 1, or otherwise, show that

$$\cos^{-1}(x) = \frac{\pi}{2} - x - \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}\frac{x^{2n+1}}{2n+1}, \text{ for } |x| < 1$$

3. Let $f_1(x) = 1$ on [0, 1], i.e., f_1 is the constant function 1 on [0, 1]. For n ≥ 2, define

$$f_n(x) = \begin{cases} nx, & 0 \le x < \frac{1}{n} \\ 2 - nx, & \frac{1}{n} \le x < \frac{2}{n} \\ 0, & \frac{2}{n} \le x \le 1 \end{cases}$$

Show that $f_n(x)$ converges to some function $f(x)$ on [0, 1] but that the convergence is not uniform.

4 Show that $f(x) = \sum_{n=0}^{\infty} e^{-nx}\cos(nx)$ converges uniformly on any subset of **R**, which is bounded below by a positive constant. Show that

$$f'(x) = -\sum_{n=0}^{\infty} ne^{-nx}[\cos(nx) + \sin(nx)] \text{ for all } x > 0.$$

5 Prove the following Ratio Test for uniform convergence.

Suppose $u_n(x)$ are bounded non-zero functions on the set S and that there exists $r < 1$ such that

$$\left|\frac{u_{n+1}(x)}{u_n(x)}\right| \le r \text{ for all } n \ge N, \text{ some integer } N \text{ and all } x \text{ in S.}$$

Then $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on S.

6. If $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$, denote its sum by $f(x)$, then show that $a_k = \frac{f^{(k)}(0)}{k!}$ for each integer $k > 0$.

7. Show that the series $\sum_{n=1}^{\infty} \frac{1}{2^n - n\sin(nx)}$ is uniformly convergent on **R**.

8. (Optional). Show that if f is continuous on [0, 1], then there is a sequence of polynomial functions $p_n(x)$ such that $f(x) = \sum_{n=1}^{\infty} p_n(x)$. [Hint: use Weierstrass Approximation Theorem.]