National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 8

1. Test the following for convergence and uniform convergence, in the respective domain..

   (i) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n}$, $0 \le x \le 1$ (ii)) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+x}$, $0 \le x < \infty$ (iii) $\sum_{n=1}^{\infty} ne^{-nx} \sin(nx)$, $x \ge a > 0$.

2. Show, by establishing the uniform convergence of the series under the integral sign on the left of each of the following statements, that the equality hold in each case.

   (i) $\int_0^{\pi}\left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\right)dx = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^3}$ (ii) $\int_1^2 \sum_{n=1}^{\infty}\left(\frac{\ln(nx)}{n^2}\right)dx = \sum_{n=1}^{\infty} \frac{\ln(4n)-1}{n^2}$

   (iii) $\int_1^2\left(\sum_{n=1}^{\infty} ne^{-nx}\right)dx = \frac{e}{e^2-1}$

   (Hint: Show that each of the series under the integral sign is dominated by a convergent constant series and apply Weierstrass M-test)

3. Show, by establishing the uniform convergence of the term by term differentiated series, each of the following.

   (i) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3}\right) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ for all real x. (ii) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{n}{x^n}\right) = -\sum_{n=1}^{\infty} \frac{n^2}{x^{n+1}}$ for $|x| > 1$.

   (iii) $\frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{1}{n^3(1+nx^2)}\right) = -2x\sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$ for all real x.

4 (i) Use Abel's Test to show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-nx}}{n}$ converges uniformly on $[0, \infty)$.

   Explain, why if $f(x) = \sum_{n=1}^{\infty}(-1)^n \frac{e^{-nx}}{n^2}$, then $f'(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{-nx}}{n}$ for $x \ge 0$.

   (ii) Use Dirichlet's Test to show that $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n+x^2}$ converges uniformly for all x on **R.**

   Explain why we can differentiate term by term to get $f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2x}{(n+x^2)^2}$

5. Knowing your theorem.

   Suppose (f_n) is a sequence of differentiable functions defined on an interval $[a, b]$. Suppose that f_n converges pointwise to a function f. Suppose each f_n' is continuous on $[a, b]$ and f_n' converges uniformly to a function g on $[a, b]$. Give justifications or reasons for the following propositions.

   (i) The function g is continuous on $[a, b]$.

   (ii) Each function f_n' and the function g are integrable on $[a, b]$.

   (iii) $\int_a^x g = \lim_{n\to\infty} \int_a^x f'_n$

   (iv) $\lim_{n\to\infty} \int_a^x f'_n = \lim_{n\to\infty} (f_n(x) - f_n(a))$

   (v) $\int_a^x g = \lim_{n\to\infty} (f_n(x) - f_n(a)) = f(x) - f(a)$

   (vi) $g = f'$

   (vii) f_n converges uniformly to f on $[a, b]$.