National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 4

1. Let A be a non-empty subset of $\mathbf{R}$. Suppose $f : A \to \mathbf{R}$ is a function. Prove that the function f is continuous at a point a in A if and only if for every sequence (a_n) in A converging to a, $(f(a_n))$ converges to $f(a)$.

2. Using question 1, show that the function $f : [0, 1] \to \mathbf{R}$ defined by
$$f(x) = \left\{ \begin{array}{ll} x, & x \text{ rational} \\ 1-x, & x \text{ irrational} \end{array} \right\}$$
is discontinuous at all x in $[0, 1]$ except at the point 1/2.

3. Suppose $f : [0, 1] \to \mathbf{R}$ is continuous and has the property that $f(x) = x^2$ for rational x. Prove that f is identically equal to x^2 .

4. Use the Bolzano-Weierstrass Theorem to prove that if f is continuous on $[a, b]$, then f is bounded on $[a, b]$. (Hint: Suppose f is unbounded. Find a sequence (a_n) such that $f(a_n) > n$ and use question 1.)

5. Suppose $\sum a_n$ and $\sum b_n$ are two series of positive terms. Suppose $a_n / b_n \to k$ and $k > 0$. Prove that $\sum a_n$ is convergent if and only $\sum b_n$ is convergent.

   Suppose $a_n / b_n \to \infty$. Prove that if $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

6. Suppose $\sum a_n$ and $\sum b_n$ are absolutely convergent , then $\sum a_n b_n$ is also absolutely convergent. Hence deduce that $\sum a_n$ converges absolutely implies that $\sum a_n^2$ is convergent. Is the converse true?

7. Using question 5 or otherwise, test the convergence of the following series:

   (i) $\sum_1^\infty \frac{1}{1+n^2}$ (ii) $\sum_1^\infty \frac{n+1}{n(n+2)}$ (iii) $\sum_1^\infty \frac{1}{n}\sin(\frac{1}{n})$

   (iv) $\sum_1^\infty \frac{1}{2n+5}$ (v) $\sum_1^\infty \left(\frac{n+1}{n^2+1}\right)^3$ (vi) $\sum_1^\infty \frac{\ln(n)}{\sqrt{n+1}}$

8. Test the following series for convergence.

   (i) $\sum_1^\infty \frac{(n!)^2}{(2n)!}$ (ii) $\sum_1^\infty \frac{n!}{n^n}$ (iii) $\sum_1^\infty \frac{n^2}{2^n}$ (iv) $\sum_1^\infty \frac{2^n}{n!}$ (v) $\sum_2^\infty \frac{1}{n\ln(n)}$

9. Determine the convergence of the following series, say whether the convergence is absolute or conditional.

   (i) $\sum_1^\infty (-1)^{n+1}\frac{n+3}{n(n+1)}$ (ii) $\sum_1^\infty (-1)^n \frac{\ln(n)}{n}$ (iii) $\sum_1^\infty (-1)^{n+1}\frac{n}{2^n}$ (iv) $\sum_1^\infty (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$, any x.

10. Investigate the convergence of $\sum_1^\infty \frac{\sin(nx)}{n^p}$ for $p > 0$.