National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 3

1. Show that ($(1+1/n)^n$) is an increasing sequence. Show that $(1+1/n)^n < 3$. Hence deduce that it is convergent.

2. (i) Suppose (a_n) is a decreasing (respectively increasing) sequence. Show that the sequence (U_n) , where $U_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$ is also a decreasing (respectively increasing) sequence. Hence deduce that the sequence ($\frac{1}{n}(1 + \frac{1}{2} + \cdots + \frac{1}{n})$) is convergent.

   (ii) Prove that $a_n \to a \Rightarrow U_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \to a$. Show that the converse is false.

3. Suppose $\underset{n\to\infty}{Lim} \frac{a_{n+1}}{a_n} = q$ and $|q| < 1$. Show that $\underset{n\to\infty}{Lim}\ a_n = 0$. Hence deduce that $\underset{n\to\infty}{Lim} \frac{x^n}{n!} = 0$ for any real number x.

4. Suppose (a_n) is a sequence defined by

   $a_1 = 1$, $a_{n+1} = 2\,(2a_n + 1)/(a_n + 3)$ for $n \geq 1$.

   Prove that $1 \leq a_n < 2$ and that (a_n) is increasing. Hence deduce that (a_n) is convergent and $a_n \to 2$.

5. Show that if for a sequence (a_n), the subsequence (a_{2n}) and (a_{2n-1}) both converges to a , then $a_n \to a$.

6. Suppose $a_1 > 0$. For $n \geq 1$, define $a_{n+1} = \frac{1}{1 + a_n}$. Show that the sequence is convergent and find its limit. [Hint: first show that (a_{2n}) and (a_{2n-1}) are bounded monotone sequences converging to the same limit.]

7. Suppose (a_n) is a monotone sequence. Prove that (a_n) is convergent if and only if (a_n^2) is convergent.

8. Suppose $a_n \to a$ and that $|a| < 1$. Show then that $a_n^n \to 0$.

9. Show that if a sequence is convergent and converges to a, then any subsequence is also convergent and converges to a .

10. If (a_n) is a Cauchy sequence and it has a convergent subsequence, then (a_n) is convergent and has the same limit as the subsequence.

11. Prove that every Cauchy sequence in **R** has a convergent subsequence. Deduce Cauchy principle of convergence for real sequencces. [Hint: Bolzano-Weierstrass Theorem.]