National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 2

1. Let $a_n = \begin{cases} 1+\frac{1}{n}, & n \text{ odd} \\ 1-\frac{1}{2^n}, & n \text{ even} \end{cases}$.

   (i) Find positive integer N_1 such that $n > N_1 \Rightarrow |a_n - 1| < 0.01$.

   (ii) Find positive integer N_2 such that $n > N_2 \Rightarrow |a_n - 1| < 0.000016$.

   (iii) Given ε in R, ε > 0, find positive integer N such that

   $$n > N \Rightarrow |a_n - 1| < \varepsilon.$$

   (Hint: Prove $2^n > n$ for any n in **N**. Let $[\,x\,]$ be the greatest integer $\le x$, i.e. $[\,x\,] = n \in \mathbf{N}$ and $n \le x < n + 1$. Then take $N = [\,1/\varepsilon\,]$.)

2. Prove that a sequence cannot converge to two different limits.

3. Prove that if $a_n \to a$, then $|a_n| \to |a|$. If $(|a_n|)$ converges , show by a counter example that (a_n) need not converge.

4. (Existence of n-th root.). Suppose $a \ge 0$ and $n \in \mathbf{N}$, prove that there is a unique b in **R**, $b \ge 0$ such that $b^n = a$. (Use the completeness property of **R**.)

   Prove by induction or otherwise, that $h > 0 \Rightarrow (1 + h)^n \ge 1 + nh$ and deduce that $a > 1 \Rightarrow 1 < a^{1/n} \le 1 + \dfrac{a-1}{n}$ and conclude that $a^{1/n} \to 1$.

   Show that $a > 1 \Rightarrow \lim\limits_{n\to\infty} a^n = +\infty$. I.e., for any $K > 0$, there exists an integer N such that $n \ge N \Rightarrow a_n > K$.

   Show that if $a_n \to +\infty$ and $a_n \neq 0$ for all n, then $1/a_n \to 0$.

   Using these results find $\lim\limits_{n\to\infty} a^n$ and $\lim\limits_{n\to\infty} a^{1/n}$ for $a = 1$, $0 < a < 1$ and $a = 0$.

5. Prove the following

   (i) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$ (ii) $\lim\limits_{n\to\infty} \dfrac{n+1}{n^3+4} = 0$.

6. Use Squeeze Theorem or the Comparison test to prove

   (i) $\lim\limits_{n\to\infty} \dfrac{\sin(n)}{n} = 0$ (ii) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$ (iii) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0$

   (iv) $\lim\limits_{n\to\infty} n^{1/n} = 1$ [Hint: write let $h_n = n^{1/n} - 1$ and show that $n = (1 + h_n)^n \ge 1 + \dfrac{n(n-1)}{2} h_n^2$]

   (v) $\lim\limits_{n\to\infty} \dfrac{\alpha(n)}{n} = 0$, where $\alpha(n)$ = number of primes dividing n. [Hint: show $\alpha(n) \le \sqrt{n}$.]

7. Show that if (a_n) converges to 0 and (b_n) is a bounded sequence, then $(a_n b_n)$ converges to 0. Hence, or otherwise, show that $\lim\limits_{n\to\infty} \left(\dfrac{2n-1}{3n+1}\right)^n = 0$.

8. Find the limit of the following sequences.

   (i) $\left(n\left(1-(1-\frac{a}{n})^{1/3}\right)\right)$, $a \le 1$ (ii) (a_n), where $a_n = \frac{1}{n^2+1} + \frac{1}{n^2+2} + \cdots + \frac{1}{n^2+n}$

   [Hint : $1 - k^3 = (1 - k)(1 + k + k^2)$. Use Squeeze Theorem.]