National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 1

" Dedekind's theory of irrational numbers, presented in his book of 1872, "Essays on the Theory of Numbers" stems from the realisation that the real number system had no logical foundation. That was when he had to give lectures on the calculus."

"Logically an irrational number is not just a single symbol or a pair of symbols, such as ratio of two integers, but an infinite collection, such as Cantor's fundamental sequence (Cauchy sequence) or Dedekind's cut. The irrational number, logically defined, is an intellectual monster, -------- "

------- *Morris Kline*

Order Axiom for R. There exists a subset P of $\mathbf{R}$ such that

(A) $a, b \in P \Rightarrow a + b \in P$.
(B) $a, b \in P \Rightarrow a \times b \in P$.
and **(C)** $a \in \mathbf{R} \Rightarrow$ precisely one of the following holds: $a \in P$, $a = 0$, or $-a \in P$.

The ordering, '>' is defined by $b > a$ if and only if $b - a$ belongs to P. Thus for any x in $\mathbf{R}$, $x > 0$ if and only if x belongs to P. $b \geq a$ if and only if $b > a$ or $b = a$.

1. Using the order axioms prove:
   (a) If $a > b$, then $a + c > b + c$ for any c.
   (b) If $a > b$ and $c > 0$, then $ac > bc$.
   (c) If $a > b$ and $b > c$, then $a > c$.
   (d) For any two elements a and b , one and only one of the following relations hold:
   $a > b, a = b, b > a$.

   Conversely, suppose that a field exists which has a relation > satisfying 1(a)- (d), show that the subset $P = \{a : a > 0\}$ satisfies the order axioms.

2. Show that in a totally ordered field $a \neq 0$ implies $a^2 > 0$. Deduce $1 > 0$. Show that the complex numbers cannot be ordered. (Hint: $i^2 = -1$).

3. Suppose $a , b \in \mathbf{R}$ and $a > b - \varepsilon$ for all $\varepsilon > 0$, ε in $\mathbf{R}$, prove that $a \geq b$

4. Let k be an upper bound for $S \subseteq \mathbf{R}$. Show that $k \in S \Rightarrow k$ is the supremum of S.

5. Suppose A is a non-empty bounded subset of $\mathbf{R}$. Let B be a non-empty subset of $\mathbf{A}$. Prove that $\sup A \geq \sup B$ and that $\inf A \leq \inf B$.

6. Find, if they exist, the supremum and infimum of each of the following subsets of the real numbers. Also decide which of these sets have the greatest element (maximum), least element (minimum).

   (i) $\{ 1/n : n \in \mathbf{Z}_+ \}$
   (ii) $\{ x : 0 \leq x \leq \sqrt{2} , x \text{ rational} \}$
   (iii) $\{ x : x^2 + x + 1 \geq 0\}$
   (iv) $\{x : x < 0 \text{ and } x^2 + x - 1 < 0 \}$
   (v) $\{ 1/n + (-1)^n : n \in \mathbf{Z}_+ \}$
   (vi) $\{ x : x = n^3 \text{ for some positive integer } n \}$

7. Let A and B be non empty bounded above subsets of $\mathbf{R}$.

   Let $A + B = \{x + y : x \in A \text{ and } y \in B \}$ Prove that $\sup (A + B) = \sup A + \sup B$.