National University of Singapore

Department of Mathematics

Level 2000 Semester 2 (2003/2004) **MA2108 Advanced Calculus II** Tutorial 10

1. Suppose Σa_n is absolutely convergent. Let Σb_n be a rearrangement of the same series. Let $U_n = ½ (|a_n|+a_n)$, $V_n = ½ (|a_n| - a_n)$, $r_n = ½ (|b_n|+b_n)$ and $s_n = ½ (|b_n| - b_n)$. Verify that these are non-negative sequences.

   (i) Show that ΣU_n and ΣV_n are convergent series with non-negative terms and that

   $a_n = U_n - V_n$ and $b_n = r_n - s_n$.

   (ii) Note that Σr_n is a rearrangement of ΣU_n and Σs_n is a rearrangement of ΣV_n. Use this, or otherwise, prove that $\Sigma r_n = \Sigma U_n$ and $\Sigma s_n = \Sigma V_n$.

   (iii) Deduce that $\Sigma b_n = \Sigma a_n$.

2. The "error function" is defined by $\mathrm{erf}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt$.

   (i) Show that erf(x) can be represented by a power series $\sum_{n=0}^{\infty} a_n x^n$ valid for all x and compute a_0 ,a_1 , a_2 , a_3 , a_4 and a_5 .

   (ii) Use part (i) to estimate the value of $\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{t^2}{2}} dt$.

3. Prove that $\sum_{n=1}^{\infty} \left(\frac{\sin(nx)}{n^3} \right) x^3$ defines a continuous function on **R**.

4. Let $f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}$ for x in [0, 1]. Show that (f_n) converges pointwise but not uniformly.

5. Can we differentiate $x = \sum_{n=1}^{\infty} \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right)$, for x in [0, 1] term by term ?

6. Show that $\sum_{n=1}^{\infty} \left(\frac{\pi}{n} - \sin(\frac{\pi}{n}) \right)$ converges.