Theorem 21′

(i) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ is absolutely convergent.

(ii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ is

Pf (i) Let $x_n = \sup\left\{ \left|\frac{a_{n+1}}{a_n}\right|, \left|\frac{a_{n+2}}{a_{n+1}}\right|, \dots \right\}$

If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$, then

$\lim_{n\to\infty} x_n = \alpha^*$. Suppose $\alpha < 1$. Then $\exists$ an integer N such that $n \geq N \Longrightarrow$

$$\alpha - \varepsilon < x_n < \alpha + \varepsilon$$

Take any c s.t $\alpha < c < 1$ and choose $\varepsilon = c - \alpha$. Then $n \geq N \Longrightarrow x_n < \alpha + \varepsilon = c$. That means $x_N = \sup\left\{\left|\frac{a_{N+1}}{a_N}\right|, \dots\right\} < c$. Hence for any $n \geq N$, $\frac{|a_{n+1}|}{|a_n|} \leq \sup\left\{\left|\frac{a_{N+1}}{a_N}\right|, \dots\right\} = x_N < c$. Thus $n \geq N \Longrightarrow |a_{n+1}| < c|a_n|$. Therefore, as in the proof of Thm 21, (i), $\sum a_n$ is absolutely convergent.

* This is the definition of $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$

(ii) Let $y_n = \inf \left\{ \left|\frac{a_{n+1}}{a_n}\right|, \left|\frac{a_{n+2}}{a_{n+1}}\right|, \cdots \right\}$.

Let $\beta = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} y_n$. If $\beta > 1$

Then let c be any number such that $\beta > c > 1$.

Thus $\beta - c = \varepsilon > 0$. Since $\lim_{n\to\infty} y_n = \beta$, $\exists$ an integer N_0 such that

$$n \geq N_0 \Rightarrow |y_n - \beta| < \varepsilon$$
$$\Rightarrow c = \beta - \varepsilon < y_n < \beta + \varepsilon$$

Hence $y_{N_0} > c$, i.e.

$$y_{N_0} = \inf \left\{ \left|\frac{a_{N_0+1}}{a_{N_0}}\right|, \left|\frac{a_{N_0+2}}{a_{N_0+1}}\right|, \cdots \right\} > c.$$

Hence for all $n \geq N_0$,

$$\left|\frac{a_{n+1}}{a_n}\right| \geq \inf \left\{ \left|\frac{a_{N_0+1}}{a_{N_0}}\right|, \cdots \right\} > c.$$

I.e. $\forall n \geq N_0$, $|a_{n+1}| > c|a_n|$.

Then as in the proof of Thm 21 (ii),

$$a_n \not\to 0$$

and so $\sum a_n$ is divergent. This completes the proof.

<u>Remark</u>. Theorem 21′ is more refined. We only need to have the $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ or $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ to apply the test. We don't require the convergence of $\left(\left|\frac{a_{n+1}}{a_n}\right|\right)$.

Example 22

(1) $\sum_{1}^{\infty} \frac{1}{n!}$ is convergent. Let $a_n = \frac{1}{n!}$

Then $\left| \frac{a_{n+1}}{a_n} \right| = \frac{1/(n+1)!}{1/n!} = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore, since the limit is less than 1, by Theorem 21 (i), $\sum \frac{1}{n!}$ is convergent.

(2) $\sum_{n=1}^{\infty} n^2 x^n$ for $x > 0$.

Let $a_n = n^2 x^n$. Then

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2 x^{n+1}}{n^2 x^n} = \left(1+\frac{1}{n}\right)^2 \cdot x$$

$$= x.$$

Thus by Th'm 21 (i) if $x < 1$, the series is convergent. If $x > 1$, the series is divergent. But for $x = 1$, the series is just $\sum n^2$ and so is divergent.

Examples of conditionally convergent series

(1) $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{n}$ is convergent by Leibnitz's Test

$a_n = \frac{1}{n} \to 0$ and is decreasing.

$\therefore \sum (-1)^{n+1} a_n = \sum (-1)^{n+1} \frac{1}{n}$ is convergent.

But $\sum a_n = \sum \frac{1}{n}$ is divergent

Note that $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2)$.

(2) $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$ is convergent by Leibnitz's Test

$a_n = \frac{1}{2n-1} \to 0$ and is decreasing

$\therefore \sum (-1)^{n+1} a_n = \sum (-1)^{n+1} \frac{1}{2n-1}$ is convergent.

But $\sum \frac{1}{2n-1}$ is divergent as

$$\frac{1}{2n-1} > \frac{1}{2n}$$

and $\sum \frac{1}{2n}$ is divergent.

Note that $\sum_{1}^{\infty} (-1) \frac{1}{2n-1} = \frac{\pi}{4}$ and the convergence is conditional.

<u>Remark</u>: 1. $\sum_{1}^{\infty} (-1)^{n+1} \frac{1}{n} = \ln(2)$ can be obtained by rewriting the n-th sum, introduce a Eulerian sequence (that converges to the Euler constant γ)

2. Both (1) + (2) can be computed by an argument in infinite power series as expansion of $\ln(1+x)$ via integration term by term and similarly $\tan^{-1}(x)$ for (2).

Our next test will make use of the Riemann integral. The test is phrased in terms of continuous function, which is always Riemann integrable on any closed and bounded interval, or which is monotonic decreasing (and hence Riemann integrable)

Theorem 23 The Integral Test.

Suppose $\sum a_n$ is a series.
Suppose the n-th term of the series can be expressed as $a_n = f(n)$, where f is a function* defined at least on the interval $[1, \infty)$ such that
(1) f is non-negative. and
(2) f is monotonically decreasing
(hence (a_n) is non-negative and decreasing).
(i). Then $\sum a_n$ converges if the sequence $\left(\int_1^n f(x)\,dx \right)$ tends to a finite limit L as $n \to \infty$. In particular the sum $\sum_{n=1}^{\infty} a_n$ lies between L and $L + a_1$
(ii) $\sum a_n$ diverges if $\int_1^n f(x)\,dx$ tends to $+\infty$ as $n \to \infty$.

* f is often stated to require continuity.

<u>Remark</u>. Note that f is non-negative implies the sequence $\left(\int_1^n f(x)\,dx\right)$ is monotonically increasing. Hence $\left(\int_1^n f(x)\,dx\right)$ is convergent iff it is bounded. We could replace the condition in (i) by stating the equivalent condition that $\left(\int_1^n f(x)\,dx\right)$ be bounded. and (i) & (ii) may be stated simply as

$$\sum a_n \text{ converges} \iff \left(\int_1^n f(x)\,dx\right) \text{ is bounded.}$$

We have assumed that f is Riemann integrable on $[1, n]$. Note that if f is monotone on $[1, n]$, f is Riemann integrable. (See for example, Cor 4 of my article "Monotone function, function of bounded variation, Fundamental Theorem of Calculus on my Calculus web site, www.math.nus.edu.sg/matngtb/Calculus.)

Thus we can drop the continuity condition in Theorem 23. Its practical aspect is to to provide some ease to compute $\int_1^n f(x)\,dx$. Availability of anti-derivative helps if the FTC can be applied here. Note also that f is non-negative implies that a_n is non-negative.

Proof. Since $f : [1, \infty) \to \mathbb{R}$ is non-negative and monotonically decreasing, f is Riemann integrable on $[1, n]$ for each $n \geqslant 1$.

Moreover for any x in $[k, k+1]$, $k \geqslant 1$

$$a_k = f(k) \geqslant f(x) \geqslant f(k+1) = a_{k+1}$$

Therefore,

$$\sum_{k=2}^{n+1} a_k = \sum_{k=2}^{n+1} f(k) = \sum_{k=2}^{n+1} f(k)\left[(k) - (k-1)\right]$$

is a Lower Riemann sum for the integral $[1, n+1]$ with partition, $1 < 2 < 3 < \cdots < n+1$.

(Why? $f(k)$ is the minimum in $[k-1, k]$

Therefore,

$$\sum_{k=2}^{n+1} a_k \leqslant \int_1^{n+1} f(x)\,dx. \qquad (1)$$

Also

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} f(k)\left[(k+1) - k\right]$$

is an Upper Riemann sum for the interval $[1, n+1]$ since $f(k)$ is the maximum in the interval $[k, k+1]$ and so

$$\int_1^{n+1} f(x)\,dx \leqslant \sum_{k=1}^{n} a_k \qquad (2)$$

Now since f is non-negative the sequence $\left(\int_1^{n+1} f(x)dx \right)$ is a monotonically increasing sequence. Therefore, $\left(\int_1^{n+1} f(x)dx \right)$ is convergent iff it is bounded.

(ii) If $\left(\int_1^{n+1} f(x)\, dx \right)$ is bounded, then by (1) $\sum_{k=2}^{n+1} a_k$ is also bounded. Consequently by the Monotone Convergence Theorem, $\sum a_k$ is convergent.

Note that if $L = \lim_{n \to \infty} \int_1^{n+1} f(x)dx$,

then from (1)

$$\sum_{k=1}^{n+1} a_k = \sum_{k=2}^{n+1} a_k + a_1 \leq \int_1^{n+1} f(x)dx + a_1$$

Therefore,

$$\sum_{k=1}^{\infty} a_k \leq L + a_1.$$

Also from (2).

$$L = \lim_{n \to \infty} \int_1^{n+1} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k.$$

Hence

$$L \leq \sum_{k=1}^{\infty} a_k \leq L + a_1$$

(ii) If $\left(\int_1^{n+1} f(x)\,dx\right)$ diverges, then since it is monotonically increasing, $\lim_{n\to\infty} \int_1^{n+1} f(x)\,dx = +\infty$ and so $\left(\int_1^{n+1} f(x)\,dx\right)$ is unbounded.

I.e. given any K, $\exists$ integer N s.t

$$n \geq N \Rightarrow \int_1^{n+1} f(x)\,dx > K.$$

Therefore, by (2),

$$n \geq N \Rightarrow \sum_{k=1}^{n} a_k \geq \int_1^{n+1} f(x)\,dx > K.$$

and so $\sum_{k=1}^{n} a_k$ is unbounded. and so is divergent. In particular

$$\sum_{k=1}^{\infty} a_k = +\infty.$$

__Example 24__

(1) $\sum_1^{\infty} \frac{1}{n^2}$ is convergent.

Let $f(x) = \frac{1}{x^2}$. Then $f(n) = \frac{1}{n^2}$. f is non-negative and decreasing. In particular

$$\int_1^n \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^n = -\frac{1}{n} + 1 \to 1$$

Hence, $\sum_1^{\infty} \frac{1}{n^2}$ is convergent and

$$1 \leq \sum_1^{\infty} \frac{1}{n^2} \leq 1 + \frac{1}{1^2} = 2.$$

by Thm 23 (i)

(2) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Here let $f(x) = \frac{1}{x}$. f is non-negative and decreasing on $[1, \infty)$. The integral

$$\int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \ln(n)$$
$$\longrightarrow +\infty.$$

Hence by Theorem 23 part (ii), $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

(3) More generally $\sum_1^{\infty} \frac{1}{n^s}$ $(s > 0)$ Converges if $s > 1$, diverges if $s \leq 1$.

Let $f(x) = \frac{1}{x^s}$ for x in $[1, \infty)$.

If $s = 1$ by (2) we know $\sum_1^{\infty} \frac{1}{n}$ is divergent

We now assume $0 < s < 1$ or $s > 1$.

Then the integral

$$\int_1^n \frac{1}{x^s}\,dx = \left[\frac{1}{-s+1} x^{-s+1}\right]_1^n$$
$$= \frac{1}{1-s}\left[n^{-s+1} - 1\right]$$

Now if $s > 1$, then $1-s < 0$ and so

$$n^{-s+1} = \frac{1}{n^{s-1}} \longrightarrow 0 \text{ as } n \to \infty;$$

since $n^{s-1} \longrightarrow +\infty$.

$\therefore$ If $s > 1$, $\int_1^n \frac{1}{x^s}\,dx$ is convergent.

Note that f is non-negative. f is decreasing since $f(x) = \frac{1}{x^s} = e^{-s\ln(x)}$

$= \frac{1}{e^{s\ln(x)}}$ is decreasing since $e^{s\ln(x)}$

is increasing for $s > 0$. Hence, by Th'm 23 (i)

$\sum_1^\infty \frac{1}{n^s}$ is convergent if $s > 1$ and

$$\frac{1}{s-1} \leq \sum_{n=1}^\infty \frac{1}{n^s} \leq \frac{1}{s-1} + \frac{1}{1^s} = \frac{1}{s-1} + 1$$

Now if $s < 1$, then $1-s > 0$. Therefore,

$$n^{-s+1} = n^{1-s} \longrightarrow +\infty \text{ as } n \to \infty.$$

Therefore, $\int_1^n \frac{1}{x^s}\,dx = \frac{1}{1-s}\left[n^{-s+1} - 1\right]$

diverges. Hence by Theorem 23 (ii)

$\sum_1^\infty \frac{1}{n^s}$ diverges if $0 < s < 1$.

<u>Example 25</u>. This is an example of the use of comparison test.

$$\sum_{k=1}^\infty \frac{k}{e^k} \text{ is Convergent.}$$

Note that for $k > 0$

$$e^k > 1 + k + \frac{k^2}{2} + \frac{k^3}{6} > \frac{k^3}{6}.$$

Thus for each positive integer k

$$\frac{e^k}{k} > \frac{k^2}{6}$$

and so $\dfrac{k}{e^k} < \dfrac{6}{k^2}$

Since these are non-negative terms and $\sum \frac{6}{k^2}$ is convergent (See e.g. Example. 24(1)) by the Comparison test (Prop. 12, pg 3.13) $\sum_{n=1}^{\infty} \frac{k}{e^k}$ is convergent.

The next test is the last in the series. It is called the root test for obvious reason. It is also a consequence of the Comparison test.

Theorem 2.6 (Root Test)

Let $\sum_1^\infty a_k$ be a series.

(i) Suppose there is a number r with

$$0 \leq r < 1$$

and there exists an integer N such that

$$k \geq N \implies |a_k|^{\frac{1}{k}} < r.$$

Then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) If there exists an integer N such that $k \geq N \implies |a_k|^{\frac{1}{k}} > r$ for some $r > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Proof. The proof is similar to that of the ratio test.

(i) $|a_k|^{\frac{1}{k}} < r \implies |a_k| < r^k$.

Thus by assumption, $n \geq N \implies |a_k| < r^k$.

Since $\sum_{k=1}^{\infty} r^k$ converges because $r < 1$, by the comparison test (Propn 12 pg 3.13), $\sum |a_k|$ is convergent and so $\sum a_n$ is absolutely convergent.

(ii) Given that $\exists$ an integer N such that $k \geq N \Rightarrow |a_k|^{\frac{1}{k}} > r \Rightarrow |a_k| > r^k$ for some $r > 1$. Since (r^k) is divergent, the above implies that $|a_k| \not\to 0$ and so $a_k \not\to 0$. Therefore, $\sum a_k$ is divergent.

This completes the proof.

Note that to show that $a_k \not\to 0$, we only need to show that a subsequence $a_{n_k} \not\to 0$. Hence We have the following refinement of Theorem 26

<u>Remark</u>

(1) If $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent

(2) If $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} > 1$, then $\sum a_n$ is divergent

<u>Proof</u> (1). If $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = r < 1$, then taking $\varepsilon = \frac{1-r}{2}$, $\exists$ integer N s.t $n \geq N_0 \Rightarrow r - \varepsilon < x_n < r + \varepsilon$

$\Rightarrow x_n < r + \frac{1-r}{2} = \frac{r+1}{2} < 1$, where $x_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \ldots\}$.

Hence for all $n \geq N_0$, $|a_n|^{\frac{1}{n}} \leq x_n < \frac{r+1}{2} < 1$. Therefore, by Theorem 26 (i), $\sum a_n$ is absolutely convergent.

(2). If $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = r > 1$, then taking $\varepsilon = \frac{r-1}{2}$ $\exists$ integer N such that $n \geq N_0 \Rightarrow r - \varepsilon < x_n < r + \varepsilon$

$\Rightarrow x_n > r - \varepsilon = r - \frac{r-1}{2} = \frac{r+1}{2} > 1$. Thus for each $n \geq N$

$x_n = \sup\{|a_n|^{\frac{1}{n}}, |a_{n+1}|^{\frac{1}{n+1}}, \ldots\} > 1$. Thus for each $n \geq N$, $\exists\ n_0 \geq n$ such $x_n \geq |a_{n_0}|^{\frac{1}{n_0}} > 1$. Hence for each $n \geq N$, $\exists\ n_0$ s.t $|a_{n_0}| > 1$. Hence $a_n \not\to 0$.

Thus $\sum a_n$ is divergent. Note that this argument works even for $\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = +\infty$

Example. $\sum_{1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is convergent.

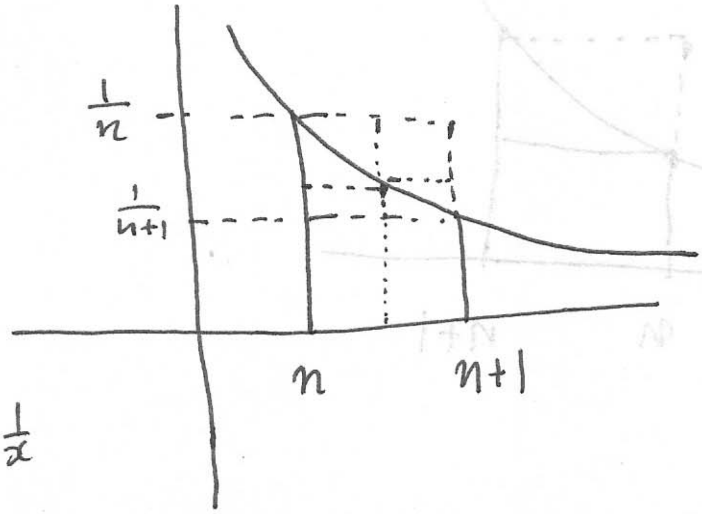
This is because,

$$\left[\left(\frac{2n+3}{3n+2}\right)^n\right]^{1/n} = \frac{2n+3}{3n+2} \to \frac{2}{3} < 1$$

as $n \to \infty$.

Therefore, by the root test, the series is convergent.

We close this chapter with a discussion of the Euler constant γ. Though the nth partial sum of $\sum \frac{1}{n}$ diverges and tends to ∞, the differen of the nth partial sum and $\ln(n)$ actually converges. It's convergence involves integration. First of all we note that $\frac{1}{n}$ and $\frac{1}{n+1}$ are respectively the maximum and minimum of $f(x) = \frac{1}{x}$ on the interval $[n, n+1]$



$f(x) = \frac{1}{x}$

Then $\frac{1}{n+1} \leq f(x) \leq \frac{1}{n}$ on $[n, n+1]$

implies that
$$\frac{1}{n+1} \leq \int_n^{n+1} f(x)dx \leq \frac{1}{n}$$

We actually have strict inequality as
$$\int_n^{n+1} f(x)dx = \int_n^{n+\frac{1}{2}} f(x)dx + \int_{n+\frac{1}{2}}^{n+1} f(x)dx$$

and so
$$\int_n^{n+1} f(x)dx \geq \frac{1}{2}\left(\frac{1}{n+\frac{1}{2}}\right) + \frac{1}{2}\left(\frac{1}{n+1}\right)$$
$$> \frac{1}{2}\left(\frac{1}{n+1}\right) + \frac{1}{2}\left(\frac{1}{n+1}\right) = \frac{1}{n+1}$$

and also
$$\int_n^{n+1} f(x)dx \leq \frac{1}{2}\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n+\frac{1}{2}}\right)$$
$$< \frac{1}{2}\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{1}{n}\right) = \frac{1}{n}.$$

Thus for each n we have
$$\frac{1}{n+1} < \int_n^{n+1} f(x)dx \leq \frac{1}{n} \qquad \text{———} \qquad (*)$$

Therefore,

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \cdots + \int_{n-1}^{n} f(x)\,dx < \sum_{k=1}^{n-1} \frac{1}{k}$$

I.e.

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \int_1^n f(x)\,dx < \sum_{k=1}^{n-1} \frac{1}{k}$$

I.e.

$$\sum_{k=2}^{n} \frac{1}{k} < \int_1^n f(x)\,dx = \int_1^n \frac{1}{x}\,dx = \ln(x)$$

$$< \sum_{k=1}^{n-1} \frac{1}{k}. \qquad \text{——— (1)}$$

Now let $s_n = \sum_{k=1}^{n} \frac{1}{k}$ for $n \geq 1$

Then from (1),

$$s_n - \ln(n) = \frac{1}{1} + \sum_{k=2}^{n} \frac{1}{k} - \ln(n)$$

$$< 1 + 0 = 1$$

and

$$s_n - \ln(n) = \frac{1}{n} + \sum_{k=1}^{n-1} \frac{1}{k} - \ln(n)$$

$$> \frac{1}{n} + 0 = \frac{1}{n}. \quad \text{for } n \geq 2$$

Let $d_n = s_n - \ln(n)$. Thus

$$0 < \frac{1}{n} < d_n < 1 \quad \text{for } n \geq 2$$

and $d_1 = s_1 - \ln(1) = 1 > 0$

Hence,

$$0 < d_n \leq 1 \quad \text{for all } n$$

and so (d_n) is bounded.

Now
$$d_{n+1} - d_n = S_{n+1} - \ln(n+1) - (S_n - \ln(n))$$
$$= (S_{n+1} - S_n) - (\ln(n+1) - \ln(n))$$
$$= \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x}\,dx$$
$$< 0 \quad \text{by } (*). \quad (p\ 3.37)$$

$\therefore \quad d_n < d_{n+1}$ for each n.

$\therefore \quad (d_n)$ is a decreasing sequence

Thus, by the Monotone Convergence Theorem (d_n) is convergent and $d_n \longrightarrow \gamma$ the so-called Euler constant.

Now we shall how we can use this constant to evaluate $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

Let $a_n = (-1)^{n+1} \frac{1}{n}$

Let now $t_n = \sum_{k=1}^{n} a_k$.

We shall rewrite t_n as follows; or rather t_{2n}.

$$t_{2n} = \sum_{k=1}^{2n} a_k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{2n} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)$$

$$= s_{2n} - s_n$$

$$= d_{2n} + \ln(2n) - (d_n + \ln(n))$$

$$= \ln(2n) - \ln(n) + d_{2n} - d_n$$

$$= \ln(2) + d_{2n} - d_n \qquad (2)$$

Since we know $\sum a_n$ is convergent (Leibnitz's Test) (t_n) is convergent and so (t_{2n}) converges to the same value as (t_n). Similar, both (d_{2n}) & (d_n) converges to the same value γ.

Thus $t_{2n} \longrightarrow \ln(2) + \gamma - \gamma = \ln(2)$.