Series Lecture 3.

Definition 1. We shall start with a sequence (a_n).

We can form the series

$$a_1 + a_2 + a_3 + \cdots$$

An (infinite) series thus consists of

(1) a sequence (a_n)

(2) the sequence (s_n) of partial sums,

where $s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$

a_n is called the <u>nth term of the series</u> and s_n is called the <u>n-th partial sum of the series</u>.

If (s_n) converges to a real number s, then we say the series converges to s and we write.

$$\sum a_n = s \quad \text{or} \quad \sum_{k=1}^{\infty} a_k = s \quad \text{or}$$

$$a_1 + a_2 + \cdots = s.$$

We usually write $\sum a_n$ or $a_1 + a_2 + \cdots$ for the series.

Example 2. The series $c + c + c + \cdots$ converges $\Leftrightarrow$ $c = 0$

and the series $0 + 0 + 0 + \cdots$ converges to 0

The series $c+c+c+\cdots$ is of course given by $\sum a_n$, with $a_n = c$ for all $n \in P$. Hence the n-th partial sum is

$$S_n = a_1 + a_2 + \cdots + a_n = nc.$$

If $c \neq 0$, then (S_n) is divergent.

Suppose $c > 0$ and $S_n \to a$. Then since $S_n = nc \geq c > 0$, $a > 0$. Given any $\varepsilon > 0$, then, $\exists$ positive integer N such that

$$n \geq N \implies |S_n - a| = |nc - a| < \varepsilon. \quad \text{--- (1)}$$

Take $\varepsilon = c$.

But by the Archimedean property of $\mathbb{R}$ $\exists$ integer N_0 such that $N_0 c > a + \varepsilon$.

Then take any $n_0 \geq \max(N, N_0)$,

$$n_0 c \geq N_0 c > a + \varepsilon > a.$$

Hence

$$|S_{n_0} - a| = |n_0 c - a| \geq |N_0 c - a|$$

$$= N_0 c - a > \varepsilon$$

This contradicts (1) since $n_0 \geq N$.

If $c < 0$, then if $S_n \to a$, $a < 0$.

Hence $-S_n \to -a$. And by the above proceeding this is not possible.

Hence (S_n) is divergent if $c \neq 0$.

Obviously, when $c = 0$ (S_n) is convergent since each S_n is equal to 0.

3.2

The above argument is trivial. It is more instructive to use the fact that any convergent sequence is bounded*. Note that if $c \neq 0$, then the n-th partial sums, S_n is not bounded. Again the use of the Archimedean property of $\mathbb{R}$ says that for any $K > 0$, $\exists$ positive integer N_0 such that $|S_{N_0}| = |N_0 c| = N_0 |c| > K$. Thus (S_n) cannot be convergent if $c \neq 0$. This is the use of the contrapositive implications. That is, $p \Rightarrow q \equiv$ not $q \Rightarrow$ not p.

That is, if a sequence is not bounded, it cannot be convergent and hence is divergent.

Example 3. $\sum_{1}^{\infty} \frac{1}{n(n+1)}$. Here $a_n = \frac{1}{n(n+1)}$.

We can write $a_n = \frac{1}{n} - \frac{1}{n+1}$.

Then $S_n = a_1 + a_2 + \cdots + a_n$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}.$$

Since $\frac{1}{n+1} \to 0$, $S_n \to 1 - 0 = 1$.

Hence $$\sum_{1}^{\infty} \frac{1}{n(n+1)} = 1$$

* Property 6. Lecture 2 Sequences.

Remark "=" sign here means much more than "=" as in $2+2=4$. It means its limit is 1. Here is an example of a new use of the equal sign.

Example 4. Geometric Series

$$\sum_{0}^{\infty} a^n = 1+a+a^2+\cdots\cdots$$

converges to $\frac{1}{1-a}$ if $|a|<1$.

Here we deviate from the previous indexing convention. We define here for a sequence a_n indexed by non-negative integers, starting with a_0, the n-th partial sum S_n to be $\sum_{k=0}^{n} a_k = a_0+a_1+a_2+\cdots+a_n$. This change in the beginning of the index will not alter theory at all. We can in effect transform this to the usual form by letting $b_k = a_{k-1}$ for $k=1,2,\cdots$, etc. Then $\sum_{k=0}^{n} a_k = \sum_{k=1}^{n+1} b_k$.

Let $c_n = a^n$, then let

$$\begin{aligned} S_n &= c_0+c_1+\cdots+c_n \\ &= 1+a+\cdots+a^n \\ &= \frac{(1+a+\cdots+a^n)(1-a)}{1-a} \quad \text{if } a\neq 1 \\ &= \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a} - \frac{a^{n+1}}{1-a}. \end{aligned}$$

Thus $s_n = \frac{1}{1-a} - \frac{c_{n+1}}{1-a}$.

Now $(c_n) = (a^n)$ converges if $|a| < 1$.

If $|a| > 1$, then $(c_n) = (a^n)$ diverges as the sequence will be unbounded.

We see on page 2(12) Example after the comparison test in lecture 2, that
$$c_n = a^n \to 0 \text{ if } |a| < 1.$$

Thus $s_n = \frac{1}{1-a} - \frac{c_{n+1}}{1-a} \to \frac{1}{1-a}$ if $|a| < 1$.

when $a = 1$, $s_n = n+1$ and the sequence (s_n) is unbounded and so (s_n) is divergent when $a = 1$. Also when $a = -1$, then
$$s_n = \begin{cases} 0, & n \text{ odd}, n \geq 1 \\ 1, & n \text{ even}, n \geq 0 \end{cases}$$
and so (s_n) is divergent.

Remark 5. Series can start from any term of a sequence (a_n).

For example. $a_0 + a_1 + \cdots$ (We start with $n = 0$)

$a_1 + a_2 + \cdots$ (start with $n = 1$)

$a_2 + a_3 + \cdots$ (start with $n = 2$)

We can write the above series as
$$\sum_{n=0}^{\infty} a_n, \quad \sum_{n=1}^{\infty} a_n, \quad \sum_{n=k}^{\infty} a_n$$

It does not matter much, does not alter theory in anyway. We can make some convention by requiring that s_n denotes the partial sum of n terms only.

Hence for example s_n can be

$$a_0 + a_1 + \cdots + a_{n-1}$$

or

$$a_1 + a_2 + \cdots + a_n$$

or

$$a_k + a_{k+1} + \cdots + a_{k+n-1}.$$

It is a simple task to verify the following:

If $k < l$, then the series

$$\sum_{n=k}^{\infty} a_n \text{ Converges} \Leftrightarrow \sum_{k=l}^{\infty} a_n \text{ Converges.}$$

Note that

$$\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{l-1} a_n + \sum_{n=l}^{\infty} a_n .$$

Thus if S_n is the nth partial sum for $\sum_{n=k}^{\infty} a_n$ and t_n is the n-th partial sum for $\sum_{n=l}^{\infty} a_n$, Then

For $n > l-k$

$$\begin{aligned} S_n &= a_k + a_{k+1} + \cdot + a_{k+n-1} \\ &= a_k + a_{k+1} + \cdots + a_l + a_{l+1} + \cdots + a_{l+(n-(l-k))-1} \\ &= a_k + a_{k+1} + \cdot + a_{l-1} + [a_l + \cdots + a_{l+(n-(l-k))-1}] \\ &= a_k + a_{k+1} + \cdots + a_{l-1} + t_{n-(l-k)} \qquad (*) \end{aligned}$$

Note that the sequence

(s_n) converges $\Leftrightarrow$ (s_{n+l-k}) converges.

(t_n) converges $\Leftrightarrow$ the sequence $(t_{n-(l-k)})$ converges.

Thus by $(*)$ t_n converges $\Leftrightarrow$ $(t_{n-(l-k)})$ converges $\Leftrightarrow$ (S_n) converges.

Properties 7

(1) If $\sum a_n$ converges, then its sum is unique.

(2) If $\sum a_n = a$ and $\sum b_n = b$, then $\sum (a_n + b_n) = a + b$.

(3) If $\sum a_n = a$, then $\sum \lambda a_n = \lambda a$.

(4) For a complex series, $\sum a_n$, converges $\Leftrightarrow$ $\sum \operatorname{Re} a_n$ + $\sum \operatorname{Im} a_n$ converges. If $\sum a_n = a$, then $\sum \operatorname{Re} a_n = \operatorname{Re} a$ and $\sum \operatorname{Im} a_n = \operatorname{Im} a$.

(1) is just an assertion about uniqueness of limits

(2) follows from (1) of Properties 8 pg 8 Lecture 2

(3) follows from (2) of Properties 8 pg 9 Lecture 2

(4) follows from the remark after the Squeeze Theorem for sequences.

Cauchy Series

We shall translate Cauchy principle of convergence for sequences to a principle of convergence for series.

Definition 8. $\sum a_n$ is a Cauchy series if the partial sum (S_n) is a Cauchy sequence, i.e., if given $\varepsilon > 0$, $\exists$ an integer N such that

$$n, m \geq N \implies |S_n - S_m| < \varepsilon. \quad \text{—(1)}$$

Suppose $n < m$. Then

$$|S_n - S_m| = |a_{n+1} + \dots + a_m| = \left| \sum_{k=n+1}^{m} a_k \right|$$

Thus (1) is equivalent to saying:
there exist integer N such that

$$m \geq n \geq N \Longrightarrow \left| \sum_{k=n+1}^{m} a_k \right| < \varepsilon .$$

or what is amount to the same thing

given $\varepsilon > 0$, $\exists$ an integer N_0 such that for all $n \geq N_0$ and for all positive integer $p >$

$$\left| \sum_{n+1}^{n+p} a_k \right| < \varepsilon .$$

<u>Theorem 3.9</u>

$\sum a_n$ is convergent $\Longleftrightarrow$ $\sum a_n$ is Cauchy.

This is just a restatement of Cauchy principle of convergence for the n-th partial sum sequence. It follows from Theorem 13 of Lecture 2. pg 19.

<u>Remark</u>. We use this to prove most of the results about series. In practice, we rarely know what the sum of the series is.

The next result is a useful means of deciding at least when a series does not converge.

Proposition 10. If $\sum a_n$ converges, then $a_n \to 0$.

Proof. If $\sum a_n$ converges, then $\sum a_n$ is Cauchy. Therefore, by Definition 8, given $\varepsilon > 0$, $\exists$ an integer N_0 such that for all $n \geq N_0$ and for all positive integer p,

$$\left| \sum_{n+1}^{n+p} a_k \right| < \varepsilon .$$

Taking $p = 1$, we have then

$$|a_{n+1}| < \varepsilon$$

for all $n \geq N_0$.

Hence $a_n \to 0$.

Remark. From proposition 10, we have if a_n does not converge to 0, then $\sum a_n$ diverges.

Example 11.

1. $\sum a^n$ diverges if $|a| \geq 1$,

since $a^n \not\to 0$ if $|a| \geq 1$.

Remark.

The converse of Proposition 10 is false.

That is, if $a_n \to 0$, $\sum a_n$ need not converge.

For example.

$a_n = \frac{1}{n} \to 0$ but $\sum a_n$ diverges.

This is because the n-th partial sum is not bounded.

We shall look at a subsequence of (s_n) by ~~s~~ successively doubling the number of terms

$$s_1 = 1,$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2},$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right), \quad \ldots,$$

we look at the sequence (s_{2^n})

Thus $S_{2^{n+1}}$ is obtained from S_{2^n} by adding the next 2^n terms

$$\left(a_{2^n+1} + a_{2^n+2} + \cdots + a_{2^n+2^n} \right)$$

$$= \frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^{n+1}}$$

$$> \frac{1}{2^{n+1}} + \cdots \cdots + \frac{1}{2^{n+1}}$$

$$= 2^n \times \frac{1}{2^{n+1}} = \frac{1}{2}.$$

So we have the following inequality

$$S_{2^{n+1}} > S_{2^n} + \frac{1}{2}.$$

Thus $S_{2^{n+1}} > S_{2^n} + \frac{1}{2} > S_{2^{n-1}} + \frac{1}{2} + \frac{1}{2}$.

$$> S_{2^{n-2}} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$> S_{2^1} + (n-1)\frac{1}{2} + \frac{1}{2}$$

$$\geq 1 + \frac{1}{2} + n\frac{1}{2} = 1 + (n+1)\frac{1}{2}.$$

Therefore (S_n) is unbounded. since for any $K > 0$, we can find an integer N such that $\frac{1}{2}N > K$ and so.

$S_{2^n} > K$ for any $n > N$.

Now we shall embark on various tests for convergence.

First we state two results about series of real non-negative terms. Therefore, the n-th partial sum is an increasing sequence.

<u>Prop^n 11.</u> Suppose $\sum a_n$ is a series of real non-negative terms. Then $\sum a_n$ converges if and only if (S_n) is bounded.

<u>Proof</u>. Since each $a_n \geq 0$,

$$S_{n+1} = a_1 + a_2 + \cdots + a_{n+1} \geq a_1 + a_2 + \cdots + a_n = S_n.$$

Hence (S_n) is an increasing sequence.

If (S_n) is bounded, then by Prop^n 10 Lecture 2 pg 16 (S_n) is convergent. On the other hand, if (S_n) is convergent, it is bounded. (Property 6 pg 12 Lecture 2)

<u>Prop^n 12.</u> (Comparison Test).

Let $\sum a_n$ and $\sum b_n$ be two series of real non-negative terms such that

$$a_n \leq \lambda b_n$$

for all n, for some non-negative real λ.

Then $\sum b_n$ converges $\Rightarrow$ $\sum a_n$ converges,
$\sum a_n$ diverges $\Rightarrow$ $\sum b_n$ diverges.

Proof: $\sum b_n$ converges $\Leftrightarrow$ $\sum b_n$ is bounded. Thus $\sum_1^n a_k \leq \lambda \sum_1^n b_k$ implies that $\sum_1^n a_k$ is bounded.
Therefore by Prop 11, $\sum a_k$ is convergent.

If $\sum a_n$ diverges, then by Prop 11, the sequence $(S_n) = \left(\sum_1^n a_k\right)$ is unbounded. Thus $t_n = \sum_1^n b_k \geq \frac{1}{\lambda} \sum_1^n a_k = \frac{1}{\lambda} S_n$ (noted that $\lambda \neq 0$) and ~~so~~ t_n is also unbounded. Thus $\sum b_n$ is divergent by Prop 11. This completes the proof.

Example 13. $\sum \frac{1}{n^2}$ is convergent:
Since $\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$ and so
$\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent since $\sum \frac{1}{n(n+1)}$ is convergent. (See Example 3). Therefore
$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ is convergent.

Propn 14. Suppose $\sum |a_n|$ converges. Then $\sum a_n$ converges.

Pf. We shall use Cauchy convergence principle for series. If $\sum |a_n|$ converges, then $\sum |a_n|$ is Cauchy. Hence given $\varepsilon > 0$, $\exists N$ s.t $n \geq N$

$$\Rightarrow \left| \sum_{k=n+1}^{n+p} |a_k| \right| = \sum_{k=n+1}^{n+p} |a_k| < \varepsilon$$

integer p.

Hence $\left| \sum_{k=n+1}^{n+p} a_k \right| \leq \sum_{k=n+1}^{n+p} |a_k| < \varepsilon$ for all $n \geq N$ and for all positive integer p.

Hence $\sum a_k$ is Cauchy. Thus by Theorem 9. $\sum a_k$ is convergent.

Remark 15. A useful equivalent statement for Propn 14 is:

If $\sum a_n$ diverges, then $\sum |a_n|$ is divergent.

<u>Defn 16</u>. We say the series $\sum a_n$ is <u>absolutely convergent</u> if $\sum |a_n|$ is convergent

The converse of Proposition 14 is false. That is a series $\sum a_n$ can be convergent but $\sum |a_n|$ need not.

<u>Example 16</u>. $\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ is convergent but not absolutely convergent.
(See remark after Example 11, pg 11.)
$\sum \frac{(-1)^{n+1}}{n}$ is convergent will be shown later by the Leibnitz's Alternating series test.

<u>Remark</u>. Most tests are tests for absolute convergence. Obviously any test about non-negative series gives a test about $\sum |a_n|$. The following is a case in point.

<u>Proposition 17</u>. Suppose (a_n) is a bounded sequence. Then $\sum \frac{a_n}{n^2}$ converges (actually absolutely.)

<u>Proof</u>: (a_n) is bounded implies that

$|a_n| \leq M$ for all $n \in P$.

Thus $0 \leq \left|\frac{a_n}{n^2}\right| \leq \frac{M}{n^2}$.

Since $\sum \frac{1}{n^2}$ converges (see Example 13 pg 14), by the Comparison Test, (Prop 12 pg 13), $\sum \left|\frac{a_n}{n^2}\right|$ is convergent. Hence by Prop 14, $\sum \frac{a_n}{n^2}$ is convergent.

<u>Example 18</u>.

$$\sum \frac{\sin(nx)}{n^2} \text{ is absolutely convergent}$$

by proposition 17. for any x in $\mathbb{R}$.

<u>Definition 19</u>. If the series $\sum a_n$ is such that $\sum a_n$ is convergent but $\sum |a_n|$ is divergent, we say $\sum a_n$ is <u>conditionally convergent</u>.

Thus the series $\sum \frac{(-1)^{n+1}}{n}$ is conditionally convergent since $\sum \left|\frac{(-1)^{n+1}}{n}\right| = \sum \frac{1}{n}$ diverges.

Now we come to the alternating series test.

<u>Prop 20. (Alternating Series Test</u>, Leibnitz's Test)

If (a_n) is a monotonic decreasing, non-negative sequence and $a_n \to 0$, then $\sum (-1)^{n+1} a_n$ converges.

<u>Proof</u>. We shall show that $\sum (-1)^{n+1} a_n$ is Cauchy.

Suppose $m > n$. Then

$$\left| \sum_{n}^{m} (-1)^{k+1} a_k \right| = \left| (-1)^{n+1} a_n + (-1)^{n+2} a_{n+1} + \cdots + (-1)^{m+1} a_m \right|$$

$$= |a_n - a_{n+1} + \cdots \cdots + (\pm) a_m|$$

$$\leq \begin{cases} |a_n - a_{n+1}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{m-1} - a_m| & \text{if } m-n \text{ is odd} \\ |a_n - a_{n+1}| + |a_{n+2} - a_{n+3}| + \cdots + |a_{m-2} - a_{m-1}| + |a_m| & \text{if } m-n \text{ is even} \end{cases}$$

$$= \begin{cases} (a_n - a_{n+1}) + (a_{n+2} - a_{n+3}) + \cdots + (a_{m-1} - a_m), & m-n \text{ odd} \\ (a_n - a_{n+1}) + \cdots \cdots + (a_{m-2} - a_{m-1}) + a_m \end{cases}$$

$$= \begin{cases} a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots - (a_{m-2} - a_{m-1}) - a_m & m-n \text{ odd} \\ a_n - (a_{n+1} - a_{n+2}) - (a_{n+3} - a_{n+4}) - \cdots - (a_{m-1} - a_m) & m-n \text{ even} \end{cases}$$

$\leq a_n$. since (a_n) is monotone decreasing.

—— (*)

Now since $a_n \to 0$, given $\varepsilon > 0$, $\exists$ integer N such that $n \geq N \implies |a_n| = a_n < \varepsilon$.
Hence from (*), for $n \geq N$ and any $m > n$

$$\left| \sum_{n}^{m} (-1)^{k+1} a_k \right| \leq a_n < \varepsilon .$$

Thus $\sum_{r} (-1)^{k+1} a_k$ is Cauchy and so is convergent by Theorem 3.9. Cauchy principle of convergence for series. This completes the proof.

<u>Alternatively</u>, we may just examine the two subsequences of the n-th partial sums consisting of (1) the odd partial sums and (2) the even partial sums.

$S_{2n+1} - S_{2n-1} = a_{2n-1} - a_{2n+1} \leq 0$. Hence $S_{2n+1} \leq S_{2n-1}$ and so (S_{2n+1}) is a decreasing sequence. Note that

$$\begin{aligned} S_{2n+1} &= a_1 - a_2 + a_3 + \cdots + - a_{2n} + a_{2n+1} \\ &= [a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n}] + a_{2n+1} \\ &\geq a_{2n+1} \geq 0 \end{aligned}$$

Hence (S_{2n+1}) is bounded below. Thus by the Monotone Convergence Theorem. (Prop 10, Lecture I, pg 16) (S_{2n+1}) is convergent.

Also $S_{2n} - S_{2n-2} = a_{2n-1} - a_{2n} \geq 0$ and so $S_{2n} \geq S_{2n-2}$ and (S_{2n}) is an increasing sequence.

Now $S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$.

Hence (S_{2n}) is bounded above. Thus, by the Monotone Convergence Theorem (S_{2n}) is also convergent.

Now $s_{2n} = s_{2n-1} - a_{2n}$ $\left(= s_{2n-1} + (-1)^{2n+1} a_{2n} \right)$

Therefore
$$\lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} - \lim_{n\to\infty} a_{2n}.$$
$$= \lim_{n\to\infty} s_{2n-1} - 0 \quad \text{since } \lim_{n\to\infty} a_n = 0$$
$$= \lim_{n\to\infty} s_{2n-1} = S$$

Thus, by Tutorial 3, Qn 5, s_n converges to S.

The next test is one of the most important test for series. D'Alembert gave the absolute convergence part of the ratio test in 1768 in Opuscules mathématiques, 5. But it was Edward Waring (1734–98) who gave in 1776 the now well known ratio test for convergence and divergence and attributed this to Cauchy.

We shall give a simplified version.

<u>Theorem 21</u> (Ratio Test, D'Alembert's Test)

Suppose the series $\sum a_n$ is such that
$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists and is equal to α.

Then (i) $\alpha < 1$ implies that $\sum a_n$ is absolutely convergent (hence convergent)

(ii) $\alpha > 1$ implies that $\sum a_n$ is divergent.

(iii) $\alpha = 1$. Then $\sum a_n$ may converge or diverge. No inference can be made. The convergence may be investigated by other methods.

Proof : (i) Suppose $\alpha < 1$. Choose a real number c such that

$$\alpha < c < 1.$$

Let $\varepsilon = c - \alpha > 0$. Then since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \alpha$, $\exists$ an integer N such that

$$n \geq N \implies \alpha - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < \alpha + \varepsilon = c$$

Therefore, for all $n \geq N$, $|a_{n+1}| < |a_n| c$.

So let p be any positive integer. Then we have.

$$|a_{N+p}| < c|a_{N+p-1}| < c^2|a_{N+p-2}| < \cdots < c^p|a_N|$$

Thus since $\sum_{p=1}^{\infty} c^p$ converges because $0 < c < 1$. (Example 4, Geometric series pg 3.4) and so by the comparison test (Propn 12 Pg. 3.13), the series $\sum_{p=1}^{\infty} |a_{N+p}|$ converges. And so by the remark on pg 3.6, $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Hence $\sum a_n$ is absolutely convergent and by Proposition 14 pg 3.15, $\sum a_n$ is convergent.

(ii). Suppose $\alpha > 1$. Choose c with $\alpha > c > 1$.

Then, as before, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \alpha$, taking $\varepsilon = \alpha - c > 0$, $\exists$ an integer N_0 such that

$$n \geq N_0 \implies \alpha - \varepsilon = c < \left|\frac{a_{n+1}}{a_n}\right| < \alpha + \varepsilon \quad \text{— } (*)$$

Then for all $n \geq N_0$, $|a_{n+1}| > c|a_n|$

Thus, $|a_{N_0+p}| > c|a_{N_0+p-1}| > \cdots > c^p |a_{N_0}|$

Since $c > 1$, the sequence $(c^p |a_{N_0}|)$ diverges. Hence $|a_{N_0+p}| \not\to 0$ as $p \to \infty$. Hence $a_{N_0+p} \not\to 0$. Then by Proposition 10 pg 3.10, $\sum_{p=1}^{\infty} a_{N_0+p}$ diverges. Therefore, $\sum a_n$ diverges.

(iii) If $\alpha = 1$, no inference can be made.

<u>Example</u>. $\sum \frac{1}{n}$ diverges when $\lim_{n\to\infty} \frac{1}{n+1} \Big/ \frac{1}{n} = 1$

$\sum \frac{1}{n^2}$ converges but $\lim_{n\to\infty} \frac{1}{(n+1)^2} \Big/ \frac{1}{n^2} = 1$

<u>Remark</u>. This is the most important test for convergence. We have actually proved a more refine version of the test since we only used one side of the inequality. $(*)$. Note $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$ iff $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \alpha$.