MA 2108 Advanced Calculus

Lecture 2. Sequences.

For most part of the discussion we shall confine to real sequences. However, most of the definitions and results apply equally well to complex sequences, i.e., sequences taking values in the complex numbers. Of course we can have sequences whose values are lying in such mathematical objects as $\mathbb{R}^n$, groups, abstract topological spaces, metric spaces, set of all functions on $[0,1]$, etc. Sequences are used for very different purposes, presenting themselves as charts of movement of share values, temperature, distances, blood pressure, etc. The purpose can be to monitor movement or fluctuation of temperature as when we are not feeling well. The attention is called when the temperature rises above 37.5°C, to seek medical attention with a physician. This is an example of a sequence approaching a certain "critical" value. Vibration problems can be approached using Fourier series, which is also a sequence. Number theory problems such as primality testing may use

Lucas sequences, which have their origin based on Fibonacci sequence. These sequences are used in the so-called N+1 primality testing, where the knowledge of the prime factors of N+1 are required to begin a search for these sequences. The goal is to find a suitable Lucas sequence such that its N+1 term is congruent to 0 modulo N. However our concern will be with the behaviour of the sequence at infinity, i.e., what happens to the values of sequence (a_n), a_n as n gets very large.

We shall look at the convergence of sequences. Through special sequences, i.e., monotone sequences, we shall study Cauchy sequences, formulate a convergence criterion (that presupposes the existence of the irrational numbers) and derive the Bolzano-Weierstrass Theorem. Although for $\mathbb{R}$, completeness is equivalent to the convergence of any Cauchy sequence, which is equivalent to that any bounded monotone sequence is convergent and which is also equivalent to the conclusion of the Bolzano Weierstrass Theorem, only the Cauchy principle of convergence is capable of generalization to $\mathbb{R}^n$ and beyond and which lead to a definition of completeness for metric spaces.

<u>Definition 1</u>. Let P be the set of positive integers. A <u>sequence</u> is simply a function from P into the set of real numbers.

P is of course $\{1, 2, 3, \dots\}$. Thus a function $a : P \longrightarrow \mathbb{R}$ is a sequence.

The image $a(n)$ is called the "nth term" of the sequence and is also written as a_n. We also write $(a_1, a_2, \dots)$ or simply (a_n) for the sequence.

Hence we use the round bracket for sequences. One should not confused $(a_1, a_2, \dots)$ with a row vector.

<u>Note</u>. Don't confuse with $\{a_1, a_2, \dots\}$ the set of points $a_1, a_2, \dots$ and $\{a_n\}$ the singleton set with just one element a_n.

<u>Remark</u>. 1. More generally if X is a set a sequence in X is a function $a : P \longrightarrow X$. For example, X can be, $\mathbb{C}$, $\mathbb{R}^2$, $\mathbb{R}^n$, $n \geq 3$, etc.

2. We can replace P by any set Q with an ordering. — a generalization of sequence called a net leading to a notion of Moore-Smith convergence of net.

Definition 2

Let (a_n) be a sequence in $\mathbb{R}$. Then we say (a_n) tends to a real number a in $\mathbb{R}$ if

for any $\varepsilon > 0$, there exists a positive integer N_0 ($\in \mathbb{P}$) such that for all n in $\mathbb{P}$, with $n \geq N_0$, we have $|a_n - a| < \varepsilon$.

I.e., $n \geq N_0 \implies |a_n - a| < \varepsilon$.

Notation:

If (a_n) tends to a, we write

$a_n \longrightarrow a$ as $n \to \infty$

or $\lim_{n \to \infty} a_n = a$

or just simply, $a_n \longrightarrow a$.

We say (a_n) converges if there exists a real number a such that $a_n \longrightarrow a$. Otherwise (a_n) diverges or is divergent.

Remark 1. The N_0 above depends on ε.

2. We may also replace "$n \geq N_0$" by "$n > N_0$" without changing the sense of definition. That is, they give rise to equivalent definitions.

Example 3.

1) $a_n = c$ $\forall n \in P$. This is a constant sequence. Therefore, $a_n \to c$.

2) $a_n = (-1)^n$. (a_n) is divergent.

why?

Because for any a in $\mathbb{R}$, we can find a $\varepsilon > 0$ such that for any N_0, there exists $n \geq N_0$ with $|a_n - a| \geq \varepsilon$.

Take $\varepsilon = 1$, then

$$|(-1) - a| + |1 - a|$$
$$\geq |(-1) - a + 1 - a| = 2$$

Hence Either $|(-1)-a| \geq 1$ or $|1-a| \geq 1$

Thus, if $|(-1)-a| \geq 1$, then for any N_0, just take any odd $n \geq N_0$ and if $|1-a| \geq 1$, take any even $n \geq N_0$.

3. $a_n = \frac{1}{n}$. Then $a_n \to 0$.

For any $\varepsilon > 0$, since $\mathbb{R}$ is archimedean, there exists a positive integer N_0 such that

$$0 < \frac{1}{N_0} < \varepsilon.$$

Thus, if $n \geq N_0$, then $0 < \frac{1}{n} \leq \frac{1}{N_0} < \varepsilon$ and that means $\left|\frac{1}{n} - 0\right| < \varepsilon$.

Since in MA1102R, we have already seen the definition of continuity of a function and properties of continuity. We shall derive the properties of convergence of sequences using continuity.

<u>Notation</u>: Let P^{-1} denotes the set $\{\frac{1}{n} : n \in P\}$.
That is $P^{-1} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
Then $P^{-1} \cup \{0\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$.

<u>Proposition 4</u>: Let (a_n) be a sequence in $\mathbb{R}$.
Define a function $f : P^{-1} \cup \{0\} \longrightarrow \mathbb{R}$
by $f(\frac{1}{n}) = a_n$ for $n \neq 0$,
and $f(0) = a$.
Then $a_n \longrightarrow a$ if and only if f is continuous at 0.

<u>Proof</u>: Recall definition of continuity at 0.
Given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that
for all $x \in P^{-1} \cup \{0\}$

$$|x - 0| < \delta \Longrightarrow |f(x) - f(0)| < \varepsilon. \quad \text{——— (1)}$$

Suppose that f is continuous at $x = 0$.
Thus given any $\varepsilon > 0$, we have $\delta > 0$ satisfying (1) above

We want to find an integer N_0 in P such that $n > N_0$ implies that $|a_n - a| < \varepsilon$.

From (1) we have

$$|x - 0| < \delta \implies |f(x) - a| < \varepsilon. \quad \text{——} (1)'$$

Let N_0 be the largest integer such that

$$N_0 \leq \frac{1}{\delta}$$

Then $n > N_0$ implies that $n > \frac{1}{\delta} \geq N_0$.

That means $\frac{1}{n} < \delta$

Thus by $(1)'$, taking x to be $\frac{1}{n}$

$$|f(\tfrac{1}{n}) - a| = |a_n - a| < \varepsilon.$$

Hence $n \geq N_0 + 1$ implies that $|a_n - a| < \varepsilon$.

$\therefore a_n \longrightarrow a$.

Conversely, suppose $a_n \longrightarrow a$.
Then given $\varepsilon > 0$, $\exists$ positive integer N_0 such that $n \geq N_0 \implies |a_n - a| < \varepsilon$.

Thus for any $x \in (-\frac{1}{N_0}, \frac{1}{N_0}) \cap (P^{-1} \cup \{0\})$

$\cdot \quad x \in P^{-1} \cup \{0\}$ and $|x| < \frac{1}{N_0}$

if $x = 0$, then $|f(x) - f(0)| = 0 < \varepsilon$

if $x \neq 0$, then $x = \frac{1}{n}$ and $\frac{1}{n} < \frac{1}{N_0}$ and so $n > \frac{1}{N_0}$ implies that $|a_n - a| < \varepsilon$,

i.e., $|f(x) - f(0)| = |f(\frac{1}{n}) - a| = |a_n - a| < \varepsilon$.

$\therefore f$ is cts at $x = 0$.

Proposition 4 allows us to formulate results about continuous functions into results about sequences.

<u>Example</u> 5.1. $\frac{1}{n^u} \to 0$ as $n \to \infty$ for all $u > 0$.

Consider $f : P^{-1} \cup \{0\} \to \mathbb{R}$

$$f\left(\tfrac{1}{n}\right) = a_n = \frac{1}{n^u} = \left(\frac{1}{n}\right)^u .$$

Thus $f(x) = x^u$. for $x \neq 0$.

$f(0) = 0$.

Recall that as a real function defined on $[0, \infty)$, $\tilde{f} : [0, \infty) \to \mathbb{R}$ is defined by $\tilde{f}(x) = x^u$, $x \neq 0$ and $\tilde{f}(0) = 0$. $\tilde{f}$ is continuous at $x = 0$.

Note that

$$\tilde{f}(x) = x^u = e^{u \ln(x)}$$

$$\lim_{x \to 0+} \tilde{f}(x) = \lim_{x \to 0+} e^{u \ln(x)} = \lim_{x \to 0+} \frac{1}{e^{-u \ln(x)}}$$

$$= 0 \quad \text{since} \quad e^{-u \ln(x)} \to +\infty \text{ as } x \to 0^+$$

$$= \tilde{f}(0)$$

Hence a_n converges to $f(0) = 0$.

2) $a_n = \frac{27n^2+3n-1}{15n^2-2n-13}$

$f : P' \cup \{0\} \longrightarrow \mathbb{R}$ is defined by

$f\left(\frac{1}{n}\right) = a_n = \frac{27n^2+3n-1}{15n^2-2n-13}$ for $n \in P$

$= \frac{27 + 3\frac{1}{n} - \frac{1}{n^2}}{15 - \frac{2}{n} - \frac{13}{n^2}}$

Thus $f(x) = \frac{27 + 3x - x^2}{15 - 2x - 13x^2}$

On $\mathbb{R}$ this is a rational function whose domain contains 0. Thus f is continuous at 0 and $f(0) = \frac{27}{15}$.

Thus $a_n \longrightarrow \frac{27}{15}$.

<u>Properties 8</u>.

1. $a_n \to a$ and $b_n \to b \Rightarrow a_n + b_n \to a+b$.

<u>Proof</u>. $f\left(\frac{1}{n}\right) = a_n$, $f(0) = a$ is cts at $x=0$

g defined by $g\left(\frac{1}{n}\right) = b_n$ $g(0) = b$ is cts at $x=0$.

Therefore $f+g$ is cts at $x=0$.

Hence $f(0)+g(0) = a+b$.

$\therefore$ $a_n + b_n \longrightarrow a+b$

since $(f+g)\left(\frac{1}{n}\right) = a_n + b_n$

and $(f+g)(0) = a+b$.

2. If $a_n \to a$, then $\lambda a_n \to \lambda a$ for any real λ.

f defined by $f(\frac{1}{n}) = a_n$ & $f(0) = a$,

$f : P' \cup \{0\} \to \mathbb{R}$

is cts at $x = 0$. Hence

λf is cts at $x = 0$.

$\therefore \lambda a_n \to \lambda f(0) = \lambda a$.

3. If $a_n \to a$, $b_n \to b$, then $a_n b_n \to ab$.

$a_n \to a$ means that $f : P' \cup \{0\} \to \mathbb{R}$ defined by $f(\frac{1}{n}) = a_n$, $f(0) = a$ and $g : P' \cup \{0\} \to \mathbb{R}$ defined by $g(\frac{1}{n}) = b_n$ & $g(0) = b$ are continuous at $x = 0$. Hence

$f \times g$ is cts at $x = 0$.

In particular $f \times g(\frac{1}{n}) = f(\frac{1}{n}) g(\frac{1}{n}) = a_n b_n$

and $f \times g(0) = f(0) g(0) = ab$.

Thus $a_n b_n \to ab$

4. Let (a_n) be a real sequence. Let a be a real no.
We want to know if $a_n \to a$.
We look at the difference $a_n - a$ and if $a_n - a \to 0$, then the answer is 'yes'.

The question is then when do we know $a_n - a \to 0$. Thus the following is a good test.

Comparison test

If there exists a sequence (b_n) such that

(i) $b_n \to 0$

(ii) $|a_n - a| \leq |b_n|$

Then $a_n \to a$.

Proof. Given $\varepsilon > 0$, $\exists$ integer N such that
$n > N \Rightarrow |b_n| < \varepsilon$.
Therefore $\forall n > N$, $|a_n - a| \leq |b_n| < \varepsilon$.
$\therefore$ $a_n \to a$.

Remark. Note that $b_n \to 0 \Rightarrow |b_n| \to 0$.
By Squeeze Theorem $a_n - a \to 0$. So this result is trivial.

Example of the use of 4.

If $|a|<1$, then the sequence (a^n) converges to 0

Since $|a|<1$, then $\frac{1}{|a|}>1$. Hence we can let $\alpha = \frac{1}{|a|}-1>0$. Then

$$\frac{1}{|a|} = 1+\alpha \text{ and } |a| = \frac{1}{1+\alpha}.$$

Hence $|a^n-0| = |a|^n = \frac{1}{(1+\alpha)^n}$.

Now $(1+\alpha)^n = 1+n\alpha +$ positive term.

$$> n\alpha$$

Hence

$$|a^n| = \frac{1}{(1+\alpha)^n} < \frac{1}{n\alpha}$$

Since $\frac{1}{n}\to 0$, $\frac{1}{n\alpha}\to 0$.

Therefore, by property 4 $a^n\to 0$.

Remark: If $a=1$, then $a^n=1$, and so $a^n\to 1$.

If $a=-1$, then $a^n=(-1)^n$ diverges. [Defn to come later, radius of convergence.]

Exercise. Complex sequence.

If $a = i$, then a^n diverges.

If $|a| > 1$, then (a^n) diverges.

We will show eventually that $a \in \mathbb{C}$

$|a| = 1$, then (a^n) converges $\iff a = 1$.

Property 5. If $a_n \to a$, $a_n, a \neq 0$, then

$a_n^{-1} \to a^{-1}$.

Proof. $a_n \to a$ implies that $f : P^{-1} \cup \{0\} \to \mathbb{R}$ is continuous at $x = 0$, where $f(\frac{1}{n}) = a_n$ and $f(0) = a$. Then since $a_n \neq 0$, $f(x) \neq 0$ ~~as for~~ $x \neq 0$ and also $f(0) = a \neq 0$. Thus since f is cts at $x = 0$, $\frac{1}{f}$ is cts at $x = 0$.

$$\frac{1}{f}(x) = \frac{1}{f(x)} = \frac{1}{a_n} = a_n^{-1}.$$

Thus $a_n^{-1} \to \frac{1}{f}(0) = \frac{1}{f(0)} = \frac{1}{a} = a^{-1}$.

Property 6. If (a_n) converges, then (a_n) is bounded.

Proof. Proof is trivial. (a_n) converges implies that $\exists\, a$ s.t $a_n \to a$. Take $\varepsilon = 1$. Then by definition of convergence, $\exists$ integer N s.t $n \geq N \implies |a_n - a| < \varepsilon$. I.e. $|a_n| < |a| + \varepsilon$

(Recall $\big||a_n| - |a|\big| \leq |a_n - a|$)

Hence the set $\{|a_n| : n \geq N\}$ is bounded by $|a|+\varepsilon = |a|+1$. Thus adding a finite no. of elements to it, $|a_1|, |a_2|, .., |a_{N-1}|$, the set remains bounded. It is thus bounded by

$$\max\{|a_1|, |a_2|, .., |a_{N-1}|, |a|+\varepsilon\} = M.$$

The converse is false.

I.e. (a_n) is bounded, does not necessarily imply that (a_n) is convergent.

<u>Example</u>. $a_n = (-1)^n$. (a_n) is bounded but not convergent.

<u>Property 7</u>. If $a_n \to a$ and $b_n \to b$ and $\exists N$ s.t $a_n \leq b_n$ $\forall n \geq N$, then $a \leq b$.

<u>Proof</u>. We can use continuity here. (See Thm 4.5.15, Introduction to Calculus.)
$a_n \to a$ & $b_n \to b$ means $f, g : P^{-1} \cup \{0\} \to \mathbb{R}$ defined by $f(\frac{1}{n}) = a_n$, $g(\frac{1}{n}) = b_n$ & $f(0) = a$, $g(0) = b$ are continuous at 0. $\lim_{x \to 0} f(x) = f(0) = a$
$\lim_{x \to 0} g(x) = g(0) = b$. Then $\forall x \in (-\frac{1}{N}, \frac{1}{N})$
$f(x) \leq g(x)$. By Theorem 4.5.15,

$$f(0) = a = \lim_{x \to 0} f(x) \leq \lim_{x \to 0} g(x) = b.$$

This property uses ordering and is a result about real sequences.

<u>Remark</u>. If $a_n \to a$ and $b_n \to b$ and $a_n < b_n$ we do not get $a < b$ necessarily.

<u>Example</u> $0 \to 0$, $\frac{1}{n} \to 0$,
$a_n = 0$ for all n, $b_n = \frac{1}{n}$, $a_n \to a = 0$, $b_n \to b = 0$.

<u>Property 5</u>. Squeeze Theorem for Sequences.
If $a_n \to a$, $b_n \to a$ and there exists an integer N such that for all $n \geq N$

$$a_n \leq c_n \leq b_n,$$

then $c_n \to a$.

<u>Pf</u>. Squeeze Theorem for functions.

<u>Remark</u>.
For a complex sequence (a_n), (a_n) converges $\iff$ the real sequence $(\mathrm{Re}\, a_n)$ and $(\mathrm{Im}\, a_n)$ converge.

Use Squeeze Theorem to deduce this

$$|\operatorname{Re} a_n| \leq |a_n| \leq |\operatorname{Re} a_n| + |\operatorname{Im} a_n|$$

Thus we have

$$|\operatorname{Re} a_n - \operatorname{Re} a| \leq |a_n - a|$$

and $|\operatorname{Im}(a_n) - \operatorname{Im} a| \leq |a_n - a|$.

Therefore, by comparison test, if $(a_n) \to a$, then $|a_n - a| \to 0$ and so $\operatorname{Re} a_n \to \operatorname{Re} a$ and $\operatorname{Im} a_n \to \operatorname{Im} a$.

Conversely since

$$0 \leq |a_n - a| \leq |\operatorname{Re} a_n - \operatorname{Re} a| + |\operatorname{Im} a_n - \operatorname{Im} a|$$

$\operatorname{Re} a_n \to \operatorname{Re} a$ and $a_n \to \operatorname{Im} a \implies a_n \to a$.

Monotone Sequences

The results here apply only to real sequences.

Definition 9. A real sequence (a_n) is increasing if $n > m \Rightarrow a_n \geq a_m$

(a_n) is a strictly increasing sequence if $n > m \Rightarrow a_n > a_m$.

(a_n) is a decreasing sequence if $n > m \Rightarrow a_n \leq a_m$.

(a_n) is a strictly decreasing sequence if $n > m \Rightarrow a_n < a_m$.

(a_n) is a monotone sequence if it is either increasing or decreasing.

Proposition 10. Suppose (a_n) is a real bounded monotonic sequence, then (a_n) is convergent.

Proof: This requires the completeness property of $\mathbb{R}$.

Suppose (a_n) is monotonic increasing. Suppose it is bounded. Then the set $S = \{a_n : n \in P\}$ is bounded above and non-empty. Therefore, by the completeness of $\mathbb{R}$, S has a supremum in $\mathbb{R}$.

Let $a = \sup\{a_n : n \in P\} = \sup S$.

We claim then that $a_n \to a$.

For any $\varepsilon > 0$, $a - \varepsilon < a$. Hence $a - \varepsilon$ is not an upperbound for S and there exists an integer N_0 s.t $a - \varepsilon < a_{N_0} \leq a$.

Since (a_n) is increasing, for all $n \geq N_0$, $a_n \geq a_{N_0}$ and so we have

$$a - \varepsilon < a_{N_0} \leq a_n \leq a.$$

Thus,
$$|a - a_n| \leq |a - a_{N_0}| < \varepsilon.$$

Therefore, by definition of convergence,

$$a_n \longrightarrow a.$$

If (a_n) is decreasing, then multiply by (-1), we have that $(-a_n)$ is increasing. $(-a_n)$ is bounded since (a_n) is. Therefore, it is bounded above. Hence by what we have just proved, $(-a_n)$ is convergent. Therefore (a_n) is convergent.

Remark. Proposition 10 can be taken as the completeness axiom for $\mathbb{R}$.

An equivalent definition for completeness for $\mathbb{R}$ is: every Cauchy sequence is convergent. The essential step in the proof of Prop 10 is to use the completeness property of $\mathbb{R}$. We can use the property "every bound monotonic sequence has a limit" to prove the completeness of $\mathbb{R}$.

Cauchy Sequence

Defn 11. (a_n) is a Cauchy sequence if and only if given $\varepsilon > 0$, there exists integer N s.t for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.

You can deduce from the definition that the points of a Cauchy sequence are closer and closer together as n, m gets larger & larger.

The following is an obvious property.

Proposition 12. If (a_n) is Cauchy, then it is bounded.

Proof. Take $\varepsilon = 1$. Then by definition, there exists an integer N such that

$$n, m \geq N \implies |a_n - a_m| < \varepsilon .$$

In particular, for all $n \geq N$, $|a_n - a_N| < \varepsilon$.

Since,

$$|a_n| - |a_N| \leq \big|\, |a_n| - |a_N| \,\big| \leq |a_n - a_N| < \varepsilon$$

we have that

$$|a_n| \leq |a_N| + \varepsilon = |a_N| + 1 .$$

Therefore (a_n) is bounded by

$$\max\{ |a_1|, \dots, |a_{N-1}|, |a_N| + \varepsilon \} .$$

Theorem 13. (Cauchy Principle of Convergence).
(a_n) is a Cauchy sequence $\iff$ (a_n) is convergent.

Proof. $(\Leftarrow)$ This is easy.

(a_n) is convergent means given $\varepsilon > 0$, $\exists N$ s.t

$$n \geq N \implies |a_n - a| < \varepsilon/2 .$$

Thus for all $n, m \geq N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Therefore, (a_n) is Cauchy.

The converse is much harder.
Suppose (a_n) is Cauchy. Then by Propz 12 (a_n) is bounded. Thus we may assume $|a_n| \leq M$ for all n for some M. We are going to use the completeness of $\mathbb{R}$.
For each $n \in P$, define the set

$$S_n = \{a_n, a_{n+1}, \dots\}.$$

This is a subset of the points of (a_n).
Then S_n is also bounded and non-empty.
Thus by the completeness of $\mathbb{R}$ both the supremum and infimum of S_n exist.
Let $x_n = \sup S_n$ and $y_n = \inf S_n$.
Note that each S_n is bounded above by M and bounded below by $-M$.
Note that for $m > n$, $S_m \subseteq S_n$.
Hence,

$$\sup S_m \leq \sup S_n$$

and

$$\inf S_m \geq \inf S_n.$$

Thus, $x_m = \sup S_m \leq x_n = \sup S_n$

and $y_m = \inf S_m \geq y_n = \inf S_n$.

This means (x_n) is a decreasing sequence and (y_n) is an increasing sequence.

Note since each $x_n \geqslant -M$, (x_n) is bounded below and $y_n \leqslant M$, (y_n) is bounded above. Therefore by Proposition 10, $x_n \longrightarrow a$ and $y_n \longrightarrow b$ for some a and b.

Now since lower bound $\leqslant$ upper bound we have that for each n.

$y_n = \inf S_n \leqslant \sup S_n = x_n$.

Therefore

$$b = \lim_{n \to \infty} y_n \leqslant \lim_{n \to \infty} x_n = a.$$

We shall prove that $a = b$.

Assuming this we shall complete the proof as follows.

Note that b is a lower bound for $\{a_n, a_{n+1}, \ldots\} = S_n$ and a is an upper bound for S_n and so for each n we have

$$y_n \leqslant a_n \leqslant x_n.$$

Therefore, by the Squeeze Theorem (Property 8) $a_n \longrightarrow a$.

Suppose $a \neq b$, then $a > b$. (since $a \geqslant b$)
We shall deduce a contradiction.
(a_n) Cauchy means given $\varepsilon > 0$ $\exists$ N_0 s.t

$$n, m \geqslant N_0 \implies |a_n - a_m| < \varepsilon.$$

We shall show that if $a<b$, then (a_n) is not Caudy. We shall show that we can find a $\varepsilon > 0$ s.t for any N_0, we can find $n, m \geqslant N_0$ with $|a_n - a_m| \geqslant \varepsilon$.

<u>Take</u> $\varepsilon = \frac{1}{3}(a-b) > 0$.

Since $x_n \downarrow a$ $\quad x_n \geqslant a > a - \varepsilon$ for any $n \in P$

Therefore, for <u>any</u> $N_0 \in P$, $x_{N_0} \geqslant a > a - \varepsilon$

Therefore $a - \varepsilon$ is not an upper bound for

$$S_{N_0} = \{a_{N_0}, a_{N_0+1}, \dots\}$$

Hence, $\exists$ $n \geqslant N_0$ such that

$$a - \varepsilon < a_n \leqslant x_{N_0} (= \sup S_{N_0}) \quad \text{---(1)}$$

Also, since $y_n \uparrow b$, for any $n \in P$

$$b \geqslant y_n.$$

Then for <u>any</u> integer $N_0 \in P$,

$$b + \varepsilon > b \geqslant y_{N_0}.$$

$\therefore$ $b + \varepsilon$ is not a lower bound for S_{N_0}

Hence, we can find $m \geqslant N_0$ s.t

$$b + \varepsilon > a_m \geqslant y_{N_0} (= \inf S_{N_0}) \quad \text{---(2)}$$

$$\therefore \quad a_n - a_m > a - \varepsilon - (b+\varepsilon) \qquad \text{from (1) + (2)}$$
$$= (a-b) - 2\varepsilon$$
$$= 3\varepsilon - 2\varepsilon = \varepsilon. \quad \text{since } a-b = 3\varepsilon$$

Hence for each $N_0 \in P$ we can find integers $n, m \geqslant N_0$ such that $|a_n - a_m| \geqslant \varepsilon$.

$\therefore$ (a_n) is not Cauchy.

This contradicts that (a_n) is Cauchy.

Hence $a = b$ and so (a_n) is convergent.

This is the most important theorem. There are a few characterization of completeness for $\mathbb{R}$ (also for $\mathbb{R}^n$). The following is one which is capable of generalization to $\mathbb{R}^n$.

Theorem 14. (Bolzano - Weierstrass Theorem).
Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

This is equivalent to every Cauchy sequence is a convergent sequence.

The basic conclusion of the theorem generalizes to complete metric spaces.

Note: A simple extension to complex sequences.

Suppose (a_n) is a complex sequence. Then (a_n) is a Cauchy sequence $\Leftrightarrow$ $(\operatorname{Re} a_n)$ and $(\operatorname{Im} a_n)$ are (real) Cauchy sequences $\Leftrightarrow$ $(\operatorname{Re} a_n)$ and $(\operatorname{Im} a_n)$ are convergent $\Leftrightarrow$ (a_n) is convergent.

This is an obvious generalization by the result:
$\mathbb{R}$ is complete $\Rightarrow$ $\mathbb{R}^n$ is complete ($\Rightarrow$ $\mathbb{C}$ is complete)

Remark.

1. (a_n) is convergent implies that (a_n) is Cauchy. This is always true in other situation when (a_n) is a rational sequence i.e. $a: P \to Q$, where Q is the rational numbers, or a normed vector space, or a metric space. But the converse need not be true. That is to say,
   (a_n) Cauchy does not necessarily implies that (a_n) is convergent. It is true if and only if Q is complete.
   For example $a: P \to \mathbb{Q}$ rational numbers
   $(a_n) = (3, 3{\cdot}1, 3{\cdot}14, \cdots\cdots)$ where
   a_n = first n digit of π.

If the converse is true, i.e., every Cauchy sequence is convergent, then we say the space Q is Complete (metric complete not order complete.)

2. Note that for $\mathbb{R}$, order complete is equivalent to metric complete. Metric complete is capable of generalization to $\mathbb{R}^n$ but not order complete. For instance the complex numbers does not have a positive cone.

Notice that our monotone convergence theorem Prop 10 is quite useful.

Recall the definition of a subsequence. Suppose (a_n) is a sequence. Then a subsequence is given by (a_{n_k}) where

$n_1 < n_2 < n_3 < \cdots$

More precisely, if $a : P \longrightarrow Q$ is a sequence a subsequence of a is given by the composition

$$P \xrightarrow{n} P \xrightarrow{a} Q$$

where n is some strictly increasing map for $P \longrightarrow P$

$$a \circ n(k) = a(n(k)) = a_{n(k)} = a_{n_k}$$

using our convention of writing the terms of a sequence. ($n(k)$ is written n_k.).

Here is a result that enables one to use Proposition 10 to prove Theorem 14, Bolzano-Weierstrass theorem.

<u>Lemma 15.</u> Every real sequence has a monotone subsequence.

<u>Proof</u>. It is sufficient to pick the so called "peak" in the graph of the function.

We say the sequence (a_n) has a <u>peak at k</u> if for all $j \geq k$, $a_k \geq a_j$. a_k is called the <u>peak</u> and k the <u>peak index</u>.

Now we shall find our subsequence by using these peaks.

If there are infinite number of these peaks, say $k_1, k_2, k_3, \ldots$ with $k_1 < k_2 < \ldots$

Then by definition of the peak

$$a_{k_1} \geq a_{k_2} \geq a_{k_3} \geq \cdots$$

Thus the subsequence (a_{k_j}) is a monotone decreasing sequence.

If there are only finite number of these peaks or no peaks, then there is an index k, beyond which there are no peaks. Let $n_1 = k+1$.

Then since n_1 is not a peak index, there exists an index n_2 s.t. $n_2 > n_1$ but $a_{n_2} > a_{n_1}$.

Similarly a_{n_2} is not a peak means, it is not true that for all $j \geq n_2$, $a_j \leq a_{n_2}$. Hence there exists an index $n_3 > n_2$ such that $a_{n_3} > a_{n_2}$.

Thus we recursively define $n_{k+1} > n_k$ such that $a_{n_{k+1}} > a_{n_k}$. Therefore, (a_{n_k}) is a monotonic increasing sequence.

Proof of Bolzano-Weierstrass Theorem.

Suppose (a_n) is a bounded sequence. Then by Lemma 15, (a_n) has a monotone subsequence (a_{n_k}). Since (a_n) is bounded, (a_{n_k}) is also bounded. Therefore, by Proposition 10, (a_{n_k}) is convergent. Hence (a_n) has a convergent subsequence.