Functions of Bounded Variation and Johnson's Indicatrix
by Ng Tze Beng

In the course of proving a change of variable theorem for the Lebesgue integral, K. G. Johnson in "*Discontinuous Functions of Bounded Variation and A New Change of Variable Theorem For A Lebesgue Integral, Duke. Math. Journal, vol 36 (1969) 117-124*" introduced an indicatrix function. We shall use this function to prove a generalization of the following result to discontinuous function of bounded variation.

> **Theorem.** Suppose g: $[a, b] \to \mathbf{R}$ is a continuous function of bounded variation. Then for any subset E such that the measure of its image under g, $m(g(E))$, is zero, we have that $m(v_g(E)) = 0$, where v_g is the total variation function of g.

We state our result as Theorem 1.

Theorem 1. Suppose g: $[a, b] \to \mathbf{R}$ is a function of bounded variation. Then for any subset E such that the measure of its image under g, $m(g(E))$, is zero, we have that $m(v_g(E)) = 0$.

We shall next describe Johnson's indicatrix function below. Note that the function is only unique up to a subset of measure zero.

Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation. Take a closed subinterval $I = [a_1, a_2]$ of $[a, b]$. Let $\{ P_i \}$ be a sequence of partitions of $I = [a_1, a_2]$ such that $P_i \subseteq P_{i+1}$ and

$$\lim_{n\to\infty} \sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \text{Total variation of } f \text{ over } I = [a_1, a_2],$$

where $P_n : a_1 = x_{0,n} < x_{1,n} < \ldots < x_{k_n,n} = a_2$ is the given partition in the sequence $\{ P_i \}$ and $\sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|$ denotes $\sum_{j=1}^{k_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|$.

For each positive integer n and $1 \le j \le k_n$, let $S_{j,n}$ be the closed interval with $f(x_{j,n})$ and $f(x_{j-1,n})$ as end points, i.e.,

$$S_{j,n} = [\, f(x_{j-1,n}),\ f(x_{j,n}),\ ] \text{ or } [f(x_{j,n}), f(x_{j-1,n})].$$

Let $\chi(S_{j,n})$ be the characteristic function of $S_{j,n}$. Then plainly $\chi(S_{j,n})$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty} \chi(S_{j,n}) = \left| f(x_{j,n}) - f(x_{j-1,n}) \right| \text{ for } 1 \le j \le k_n .$$

Corresponding to each partition P_n, let

$$T_n = \sum_{j=1}^{k_n} \chi(S_{j,n}).$$

Then T_n is measurable. In particular,

$$\int_{-\infty}^{\infty} T_n(y)dy = \sum_{j=1}^{k_n} \int_{-\infty}^{\infty} \chi(S_{j,n}) = \sum_{j=1}^{k_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| = \sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right| .$$

Since P_{n+1} refines P_n, it can be easily shown that $T_{n+1}(y) \ge T_n(y)$. Then $\{ T_n \}$ is an increasing sequence of non-negative Lebesgue integrable (hence measurable) functions.

We now define with respect to this sequence of partition $\{ P_i \}$ for I,

$$T_I = T_{[a_1, a_2]} = \lim_{n\to\infty} T_n .$$

Then T_I is Lebesgue integrable or summable and by the Monotone Convergence Theorem,

$$\int_{-\infty}^{\infty} T_I(y)dy = \lim_{n\to\infty} \int_{-\infty}^{\infty} T_n(y)dy = \lim_{n\to\infty} \sum_{P_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|$$
$$= \text{Total variation of } f \text{ over } I = [a_1, a_2],$$

Definition 2. Following K.G. Johnson, we define the indicatrix of f_I, the restriction of f to the subinterval $f|_I$ to be T_I. Note that T_I is not unique, it depends on the sequence of partitions $\{ P_n \}$ used. However, T_I is unique upto a subset of measure zero. That is to say, if we have obtained T_I' using another sequence of partitions $\{ Q_n \}$, then $T_I' = T_I$ almost everywhere.

Remark. Note that $\int_{-\infty}^{\infty} T_I(y)dy$ = Total variation of f over I so long as f is of bounded variation. So the equality applies to discontinuous function of bounded variation, whereas for the Banach indicatrix function N, for discontinuous function of bounded variation, $\int_{-\infty}^{\infty} N_I(y)dy$ = the total variation of f on I − the sum of all the saltuses of f on I.

Proposition 3. T_I is unique up to a subset of measure zero. That is to say, if the sequence of partitions $\{ P_i \}$ is used to define the indicatrix function $T_{I(P)}$ and the sequence of partitions $\{ Q_i \}$ for I is used to define the indicatrix function $T_{I(Q)}$, then $T_{I(P)} = T_{I(Q)}$ almost everywhere.

Proof. Let $\{ R_n \}$ be the sequence of common refinement for $\{ P_n \}$ and $\{ Q_n \}$. We can take $R_n = P_n \cup Q_n$. Let $T_{I(R)}$ be the indicatrix function defined by $\{ R_n \}$. Then $T_{I(R)} = \lim_{n\to\infty} T_{I(R),n} = \lim_{n\to\infty} \sum_{R_n} \left| f(x_{j,n}) - f(x_{j-1,n}) \right|$ and $\int_{-\infty}^{\infty} T_{I(R)}(y)dy$ = Total variation of f over I (= $[a_1, a_2]$) and is equal to $v_f(a_2) - v_f(a_1)$, where v_f is the total variation function of f. Also, since R_n is a refinement of both P_n and Q_n,

$$T_{I(R),n}(y) \geq T_{I(P),n}(y) \text{ and } T_{I(R),n}(y) \geq T_{I(Q),n}(y).$$

Thus passing to the limit we have,

$$T_{I(R)}(y) \geq T_{I(P)}(y) \text{ and } T_{I(R)}(y) \geq T_{I(Q)}(y).$$

We now claim that $T_{I(R)} = T_{I(P)}$ almost everywhere. We show this by way of contradiction. Suppose there exists a set of measure > 0 such that $T_{I(R)}(y) > T_{I(P)}(y)$ for y in this set. Then $\int_{-\infty}^{\infty} T_{I(R)}(y)dy > \int_{-\infty}^{\infty} T_{I(P)}(y)dy$. But $\int_{-\infty}^{\infty} T_{I(R)}(y)dy = \int_{-\infty}^{\infty} T_{I(P)}(y)dy$ = Total variation of f over I. This contradiction shows that $T_{I(R)} = T_{I(P)}$ almost everywhere. Similarly, we show that $T_{I(R)} = T_{I(Q)}$ almost everywhere and so $T_{I(P)} = T_{I(Q)}$ almost everywhere.

Our next result is a technical lemma, which says that the indicatrix function over the whole of the interval $[a, b]$ dominates the sum of indicatrix functions over a countable (finite or denumerable) sequence of disjoint closed intervals in $[a, b]$.

Lemma 4. Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation and $\{ I_i \}$ is a sequence of pairwise disjoint closed intervals, each a subset of $I = [a, b]$. Then

$$T_I(y) \geq \sum_i T_{I_i}(y) \quad \text{almost everywhere.}$$

Proof. We prove the inequality for a finite collection of disjoint closed intervals, $\{ I_1, I_2, I_3, \ldots, I_k \}$. Note that any union of the finite collection of partitions of $\{ I$

$_1, I_2, I_3, \dots, I_k\}$ is a subset of a partition of $I = [a, b]$ since the members of the collection $\{I_1, I_2, I_3, \dots, I_k\}$ are disjoint closed intervals. Take typical sequences of partitions for I and for $\{I_1, I_2, I_3, \dots, I_k\}$ for definition of the indicatrix functions. Refine the sequence of partitions for I to include the partitions for I_1, I_2, $I_3, \dots$ and I_k. Denote the sequence of partitions for I by $\{R_n\}$ and the sequence of partitions for I_j, by $\{P_{j,n} : n = 1, \dots\}$. Then we have

$$T_{I(R),n}(y) \geq \sum_{j=1}^{k} T_{I_j(P_j),n}(y) \ .$$

Thus, passing to the limit we have,

$$T_I(y) \geq \sum_{j=1}^{k} T_{I_j}(y) \quad \text{almost everywhere.}$$

Therefore, for a sequence $\{I_i\}$ of pairwise disjoint closed intervals in $[a, b]$,

$$T_I(y) \geq \lim_{k \to \infty} \sum_{j=1}^{k} T_{I_j}(y) = \sum_{j=1}^{\infty} T_{I_j}(y) \quad \text{almost everywhere.}$$

The next result is a trivial consequence of the definition of the indicatrix function.

Lemma 5. Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation and I is a closed interval in $[a, b]$. Suppose

$$y \notin [\inf\{f(x) : x \in I\}, \ \sup\{f(x) : x \in I\}].$$

Then $T_I(y) = 0$.

Lemma 6. Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation and $I = [a_1, a_2] \subseteq [a, b]$. Let v_f be the total variation function of f, *i.e.*, $v_f(t)$ = total variation of f on $[a, t]$ for t in $[a, b]$. Then

$$m^*(v_f(I)) \leq \int_{-\infty}^{\infty} T_I(y)dy,$$

where m^* is the Lebesgue outer measure.
If f is also continuous, the inequality becomes an equality.

Proof. $m^*(v_f(I)) = m^*(v_f([a_1, a_2])) \leq v_f(a_2) - v_f(a_1)$
= total variation of f on I,
$= \int_{-\infty}^{\infty} T_I(y)dy.$

If f is also continuous, then v_f is also continuous and increasing and so

$$v_f(I) = [v_f(a_1), \ v_f(a_2)].$$

Consequently,

$$m^*(v_f(I)) = m^*(v_f([a_1, a_2])) = v_f(a_2) - v_f(a_1) = \int_{-\infty}^{\infty} T_I(y)dy.$$

Lemma 7. Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation and $\{I_j\}$ is a sequence of pairwise disjoint closed intervals, each a subset of $I = [a, b]$. Let $S = \bigcup_j I_j$, the union of all the I_j's. Suppose A is a measurable subset of $\mathbf{R}$ such that $[\inf\{f(x) : x \in I_j\}, \ \sup\{f(x) : x \in I_j\}] \subseteq A$ for each j. Then

$$m^*(v_f(S)) \leq \int_A \sum_{j=1} T_{I_j}(y)dy \leq \int_A T_I(y)dy \ .$$

Proof. $m^*(v_f(S)) \leq \sum_{j=1}^{\infty} m^*(v_f(I_j)) \leq \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} T_{I_j}(y)dy$, by Lemma 6,

$$\leq \sum_{j=1}^{\infty} \int_A T_{I_j}(y)dy = \int_A \sum_{j=1}^{\infty} T_{I_j}(y)dy, \text{ by Lemma 5,}$$
$$\leq \int_A T_I(y)dy \text{ , by Lemma 4.}$$

We shall need also the following result concerning the measure of a union of a denumerable collection of subsets of $[a, b]$.

Lemma 8. Suppose $A_1, A_2, \ldots$ is a sequence of subsets of $[a, b]$. Then there exists an integer k such that

$$m^*(\bigcup_{n=1}^{k} A_n) \geq \frac{1}{2} m^*(\bigcup_{n=1}^{\infty} A_n),$$

where m^* denotes the Lebesgue outer measure**.**

Proof. If $m^*(\bigcup_{n=1}^{\infty} A_n) = 0$**,** we have nothing to prove since both sides of the inequality is zero. If $m^*(\bigcup_{n=1}^{\infty} A_n) > 0$, its just an exercise in the convergence of sequence. Since the Lebesgue outer measure is regular,

$$m^*(\bigcup_{n=1}^{\infty} A_n) = \lim_{j \to \infty} m^*(\bigcup_{n=1}^{j} A_n).$$

Since $\bigcup_{n=1}^{\infty} A_n \subseteqq [a,b]$**,** $0 < m^*(\bigcup_{n=1}^{\infty} A_n) \leq b - a < \infty$**,** the limit is finite. By the definition of limit, there exists an integer k such that for all $j \geq k$,

$$\left| m^*(\bigcup_{n=1}^{j} A_n) - m^*(\bigcup_{n=1}^{\infty} A_n) \right| < \frac{1}{2} m^*(\bigcup_{n=1}^{\infty} A_n).$$

Hence, $m^*(\bigcup_{n=1}^{k} A_n) > \frac{1}{2} m^*(\bigcup_{n=1}^{\infty} A_n)$. Thus, there exists an integer k such that

$$m^*(\bigcup_{n=1}^{k} A_n) \geq \frac{1}{2} m^*(\bigcup_{n=1}^{\infty} A_n).$$

Theorem 9. Suppose $f : [a, b] \to \mathbf{R}$ is a function of bounded variation. Suppose E is a subset of $[a, b]$ such that f is continuous at each point of E and that the measure of its image under f, $m(f(E))$, is zero. Then $m(v_f(E)) = 0$.

Proof. Since $m(f(E)) = 0$, for each positive integer n there exists an open set A_n such that $f(E) \subseteq A_n$ and $m(A_n) \leq 1/n$. For each e in E, f is continuous at e and $f(e) \in A_n$. Therefore, there exists $\varepsilon > 0$ such that $(f(e) - \varepsilon, f(e) + \varepsilon) \subseteq A_n$. Then there exists $\delta(e) > 0$ such that

$$f((e - \delta(e), e + \delta(e))) \subseteq (f(e) - \varepsilon/2, f(e) + \varepsilon/2) \subseteq A_n.$$

Note that $f([e - \delta(e)/2, e + \delta(e)/2])) \subseteq (f(e) - \varepsilon/2, f(e) + \varepsilon/2) \subseteq A_n$. Let $I_e = (e - \delta(e)/2, e + \delta(e)/2)$. Then

$$f(\overline{I_e}) = f([e - \delta(e)/2, e + \delta(e)/2]) \subseteqq [\inf f(\overline{I_e}), \sup f(\overline{I_e})]$$
$$\subseteqq [f(e) - \varepsilon/2, f(e) + \varepsilon/2] \subseteqq (f(e) - \varepsilon, f(e) + \varepsilon) \subseteqq A_n.$$

The collection $\{ I_e ; e \in E\}$ is an open cover for E. Therefore, by Lindelöf Theorem, there exists a countable subcover $\{ I_1, I_2, I_3, \ldots \}$ for E.
We claim that

$$m^*(v_f(\bigcup_{i=1}^{\infty} I_i)) \leq 2 \int_{A_n} T_I(y)dy, \text{ ------------------------- (1)}$$

where $I = [a, b]$.
By Lemma 8, $\frac{1}{2} m^*(v_f(\bigcup_{i=1}^{\infty} I_i))) \leq m^*\big(v_f(\bigcup_{i=1}^{k} I_i)\big)$ for some positive integer k.
Thus,

$$m^*(v_f(\bigcup_{i=1}^{\infty} I_i))) \leq 2m^*\big(v_f(\bigcup_{i=1}^{k} I_i)\big). \text{ ------------------ (2)}$$

Note that $\bigcup_{i=1}^{k} \overline{I_i}$ is a finite union of closed interval and so it is a disjoint union of closed interval say, $C_1, C_2, C_3, \ldots, C_N$. In particular, note that each C_j is connected and is a finite union of members $\{\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_k\}$, where the union cannot be partitioned into two disjoint collections, so the corresponding collections

$$\{[\inf f(\overline{I}_i), \sup f(\overline{I}_i)], i = 1, 2, \ldots, k\}$$

inherits the same property that the union cannot be partitioned into two disjoint collections. It follows then, since each $[\inf f(\overline{I}_j), \sup f(\overline{I}_j)] \subseteqq A_n$,

$$[\min_{1\le i\le k} \inf f(\overline{I}_i), \max_{1\le i\le k} \sup f(\overline{I}_i)] = [\inf f(C_j), \sup f(C_j)] \subseteqq A_n .$$

Then by Lemma 7,

$$m^*\big(v_f(\bigcup_{i=1}^{k} I_i)\big) \le m^*\big(v_f(\bigcup_{i=1}^{k} \overline{I}_i)\big) \le m^*\big(v_f(\bigcup_{i=1}^{N} C_i)\big) \le \int_{A_n} T_I(y)dy.$$

It then follows from (2) that

$$m^*(v_f(\bigcup_{i=1}^{\infty} I_i)) \le 2\int_{A_n} T_I(y)dy.$$

Since $E \subseteqq \bigcup_{i=1}^{\infty} I_i$, $m^*(v_f(E)) \le m^*(v_f(\bigcup_{i=1}^{\infty} I_i)) \le 2\int_{A_n} T_I(y)dy$. It follows that $m^*(v_f(E)) \le 0$ because $m(A_n) \to 0$ as $n\to\infty$ so that $\lim_{n\to\infty} \int_{A_n} T_I(y)dy = 0$. (Apply for instance, the Lebesgue Dominated Convergence Theorem.) This means $m^*(v_f(E)) = 0$.

Proof of Theorem 1.

Since $g : [a, b] \to \mathbf{R}$ is a function of bounded variation, its set of discontinuity D is at most denumerable. Note that then $m(g(D)) = m(v_g(D)) = 0$, since the image set $g(D)$ and $v_g(D)$ are at most denumerable. Suppose a subset E is such that $m(g(E)) = 0$. Then $m(g(E - D)) = 0$ and g is continuous at every point of $E - D$. Therefore, by Theorem 9, $m(v_g(E-D)) = 0$. It follows that

$$m^*(v_g(E)) \le m^*(v_g(E-D)) + m^*(v_g(E\cap D)) = 0 + 0 = 0.$$

Hence, $m(v_g(E)) = 0$.

Some properties of monotone functions

Lemma 10. Suppose $f : [a, b] \to \mathbf{R}$ is a monotone increasing function and E a subset of $[a, b]$. Then we can write $f = g + h$, where g is an absolutely continuous increasing function on $[a, b]$ and h is an increasing singular function on $[a, b]$. (See Theorem 15 of my article, "*Arc Length, Functions of Bounded Variation and Total Variation*" .) Then

$$m^*(f(E)) \ge m^*(g(E)),$$

where m^* is the Lebesgue outer measure. If E is measurable, then we have

$$m^*(f(E)) \ge m(g(E)),$$

where m is the Lebesgue measure.

Proof. Since $m^*(f(E))$ is finite, given $\varepsilon > 0$, there exists an open set V such that $f(E) \subseteq V$ and

$$m^*(V) < m^*(f(E)) + \varepsilon\,. \text{ -------------------------- } (1)$$

Since V is open, it is a union of countable (finite or denumerable) disjoint open intervals. That is, $V = \bigcup B_k$, where each B_k is an open interval. Since f is measurable, each $f^{-1}(B_k)$ is measurable and the collection $\{f^{-1}(B_k) : k = 1,2, \ldots\}$ is a collection of disjoint measurable subsets in $[a, b]$ and $\bigcup_k f^{-1}(B_k) \supseteqq E$.

We claim that $m^*(g(f^{-1}(B_k))) \le m(B_k)$. We show this below.

Suppose β and γ are in $g(f^{-1}(B_k))$ such that $\beta > \gamma$. Then there exist x and y in B_k such that $\beta = g(f^{-1}(x))$ and $\gamma = g(f^{-1}(y))$. Since f and g are increasing, $x > y$. Then

$$\beta - \gamma = g(f^{-1}(x)) - g(f^{-1}(y)) = f(f^{-1}(x)) - h(f^{-1}(x)) - (f(f^{-1}(y)) - h(f^{-1}(y)))$$
$$= x - h(f^{-1}(x)) - (y - h(f^{-1}(y))) = x - y - (h(f^{-1}(x)) - h(f^{-1}(y)))$$
$$\leq x - y \leq \text{diameter of } B_k.$$

Since this is true for any β and γ in $g(f^{-1}(B_k))$, we conclude that the diameter of $g(f^{-1}(B_k)$ ≤ diameter of B_k. Hence,

$$m^*(g(f^{-1}(B_k))) \leq \text{diameter of } B_k = m(B_k). \quad \text{------------------} \quad (2)$$

Therefore,

$$m^*(g(E)) \leq m^*\left(\bigcup_k g(f^{-1}(B_k))\right) \leq \sum_k m^*(g(f^{-1}(B_k)))$$
$$\leq \sum_k m(B_k) = m(V), \text{ by (2)},$$
$$< m^*(f(E)) + \varepsilon, \text{ by (1)}.$$

Since ε is arbitrarily small,

$$m^*(g(E)) \leq m^*(f(E)).$$

If E is measurable, since g is absolutely continuous, $g(E)$ is measurable and so $m^*(g(E)) = m(g(E))$.

Theorem 11. Suppose $f : [a, b] \to \mathbf{R}$ is a monotone increasing absolutely continuous function and E is a measurable subset of $[a, b]$. Then

$$\int_E f'(x)dx = m(f(E)).$$

Proof. We begin by proving the theorem for the special case when E is an open subset of $[a, b]$. Since E is open, E = a countable (finite or denumerable) union of *disjoint* open intervals, say $\{U_1, U_2, \ldots\}$. Thus

$$m(f(E)) = m(f(\bigcup_n U_n)) = m(\bigcup_n f(U_n)) = \sum_n m(f(U_n)),$$

since $\{f(U_1), f(U_2), \ldots\}$ is a collection of non-overlapping intervals,

$$= \sum_n (f(b_n) - f(a_n)), \text{ where } U_n = (a_n, b_n),$$
$$= \sum_n \int_{a_n}^{b_n} f'(x)dx, \text{ because } f \text{ is absolutely continuous},$$
$$= \int_E f'(x)dx.$$

Note that since E is measurable and f is absolutely continuous, $f(E)$ is measurable so that $m^*(f(E)) = m(f(E))$. Also, for any open U, $f(U)$ is measurable and so $m^*(f(U)) = m(f(U))$.

For the general case, suppose now E is a measurable subset in $[a, b]$. Then for each positive integer n, there exists an open set G_n, such that $E \subseteq G_n$ and $m(G_n) < m(E) + 1/n$ and an open set H_n, such that $f(E) \subseteq H_n$ and $m(H_n) < m(f(E)) + 1/n$. Thus, $\lim_{n\to\infty} m(G_n) = m(E)$ and $\lim_{n\to\infty} m(H_n) = m(f(E))$.

For each positive integer n, $f^{-1}(H_n)$ is open by continuity of f. Therefore, $C_n = f^{-1}(H_n) \cap G_n$ is also open and contains E. Note that

$$m(E) \leq \lim_{n\to\infty} m(C_n) \leq \lim_{n\to\infty} m(G_n) = m(E).$$

Hence, $\lim_{n\to\infty} m(C_n) = m(E)$.

Similarly, since $f(E) \subseteq f(C_n) = f(f^{-1}(H_n) \cap G_n) \subseteq H_n$,

$$m(f(E)) \leq \lim_{n\to\infty} m(f(C_n)) \leq \lim_{n\to\infty} m(H_n) = m(f(E)).$$

Thus, $\lim_{n\to\infty} m(f(C_n)) = m(f(E))$.

Therefore,

$$m(f(E)) = \lim_{n\to\infty} m(f(C_n)) = \lim_{n\to\infty} \int_{C_n} f'(x)dx\,, \text{ since } C_n \text{ is also open,}$$
$$= \int_E f'(x)dx\,, \text{ by Lebesgue Dominated Convergence Theorem.}$$

This completes the proof.

If f is monotone increasing but not necessarily absolutely continuous, we have the following result.

Theorem 12. Suppose $f:[a, b] \to \mathbf{R}$ is a monotone increasing function and C is a measurable subset of $[a, b]$. If E is the subset of C where f' exists (finitely), then

$$\int_C f'(x)dx = m^*(f(E)) \le m^*(f(C)).$$

Proof. Note that since $f:[a, b] \to \mathbf{R}$ is a monotone increasing function, f is differentiable almost every where. Thus, $m(C-E) = 0$.

First we note the following result.

By Theorem 2 of my article, "*Functions Having Finite Derivatives, Bounded Variation, Absolute Continuity, the Banach Zarecki Theorem and de La Vallée Poussin's Theorem*",

$$m^*(f(E)) \le \int_E |f'(x)|dx = \int_E f'(x)dx = \int_C f'(x)dx\,. \text{ ------------ } (1)$$

By Lemma 10, we can decompose f as a sum $f = g + h$, where g is absolutely continuous and increasing and h is an increasing singular function on $[a, b]$ and

$$m^*(f(E)) \ge m(g(E)).$$

Since g is monotone increasing and absolutely continuous, by Theorem 11,

$$m(g(E)) = \int_E g'(x)dx.$$

But

$$\int_E g'(x)dx = \int_E f'(x)dx\,, \text{ since } g' = f' \text{ almost everywhere,}$$
$$= \int_C f'(x)dx.$$

Hence, $m^*(f(C)) \ge m^*(f(E)) \ge m(g(E)) = \int_C f'(x)dx.$

Corollary 13. Suppose $f:[a, b] \to \mathbf{R}$ is a function of bounded variation and E is a measurable subset of $[a, b]$. Then

$$m^*(v_f(E)) \ge \int_E |f'(x)|dx.$$

If f is absolutely continuous, the inequality becomes an equality.

Proof. By Theorem 12,

$$m^*(v_f(E)) \ge \int_E v_f'(x)\,dx.$$

Since f is of bounded variation, $|f'(x)| = v_f'(x)$ almost everywhere and so

$$\int_E v_f'(x)dx = \int_E |f'(x)|dx$$

and $$m^*(v_f(E)) \ge \int_E |f'(x)|dx.$$

If f is absolutely continuous, then v_f is also absolutely continuous and since it is increasing, by Theorem 11,

$$m^*(v_f(E)) = \int_E v_f'(x)\,dx = \int_E |f'(x)|dx\,.$$

Theorem 14. Suppose f :[a, b] $\rightarrow$ **R** is a monotone increasing function and E is the subset of [a, b], where f' exists finitely. Then

$$\int_a^b f'(x)dx = m^*(f(E)) \leq f(b) - f(a).$$

Proof. By Theorem 12,

$$\int_a^b f'(x)dx = \int_{[a,b]} f'(x)dx = m^*((f(E)) \leq f(b) - f(a).$$

We now apply our results to prove a weaker version of Theorem 2 in my article "*Change of Variables Theorem*".

Theorem 15. Suppose f :[a, b] $\rightarrow$ **R** is a function of bounded variation and E is a measurable subset of [a, b] such that $m(f(E)) = 0$. Then $f' = 0$ almost everywhere on E.

Proof. By Theorem 1, $m(v_f(E)) = 0$. By Corollary 13, since v_f is monotone increasing,

$$m^*(v_f(E)) \geq \int_E |f'(x)|dx.$$

Plainly, $m^*(v_f(E)) = m(v_f(E)) = 0$ and so $\int_E |f'(x)|dx = 0$. This implies that $f' = 0$ almost everywhere on E.

We close this article with the converse to Theorem 1.

Theorem 16. Suppose g: [a, b] $\rightarrow$ **R** is a function of bounded variation. Then for any subset E such that the measure of its image under v_g, $m(v_g(E))$, is zero we have that $m(g(E)) = 0$.

Proof.
Since $m(v_g(E)) = 0$, given $\varepsilon > 0$, there exists an open set U such that $U \supseteq v_g(E)$ and $m^*(U) < \varepsilon$. Since U is open, U is a disjoint union of countable number of open intervals I_i, $i = 1, ..., n$, i.e., $U = \bigcup_{i=1}^{\infty} I_i$ and $m^*(U) = \sum_{i=1}^{\infty} m^*(I_i) < \varepsilon$. Then $v_g^{-1}(U) \supseteq E$.
Let $A_i = g(v_g^{-1}(I_i))$. For any x and y in A_i, there exists a, b in $v_g^{-1}(I_i)$ such that $x = g(a)$ and $y = g(b)$. Therefore,

$$|x - y| = |g(a) - g(b)| \leq |v_g(a) - v_g(b)| \leq m^*(I_i).$$

It follows that Diameter $A_i \leq m^*(I_i)$ and so $m^*(A_i) \leq m^*(I_i)$.
Note that $g(E) \subseteq g(v_g^{-1}(U)) = g\left(v_g^{-1}\left(\bigcup_{i=1}^{\infty} I_i\right)\right)$. Hence,

$$m^*(g(E)) \leq m^*\left(g\left(v_g^{-1}\left(\bigcup_{i=1}^{\infty} I_i\right)\right)\right) \leq \sum_{i=1}^{\infty} m^*\left(g\left(v_g^{-1}(I_i)\right)\right) = \sum_{i=1}^{\infty} m^*(A_i) \leq \sum_{i=1}^{\infty} m^*(I_i) < \varepsilon.$$

Since ε is arbitrary, we conclude that $m^*(g(E)) = 0$.

Ng Tze Beng
November 2009